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FOUNDATIONS

Scroggs, Schiller Joe. Extensions of the Lewis system S5. J. Symbolic Logic 16, 112–120 (1951).

This paper treats of matrices satisfying $S5$ and extensions of $S5$. A Henle matrix is defined as a normal $S5$-matrix with $^*x=1$ for $x\neq 0$ (* corresponds to $\diamond$). Every finite Henle matrix is equivalent to a matrix H_n, where H_n is the boolean algebra of 2^n elements with one designated element 1 and $^*0=0$, $^*x=1$ for $x\neq 0$. Moreover, every finite normal $S5$-matrix is equivalent to some H_n. An extension of $S5$ is called quasi-normal if it is closed under substitution and preserves the rule of detachment under material implication. If S is a quasi-normal extension of $S5$ and the formula α is not provable in S, then α is not satisfied by some finite normal S-matrix, so there is an index n for which H_n does not satisfy α. On the other hand, if S is consistent and not identical with $S5$, there is a greatest integer m such that H_m is an S-matrix; it follows that H_m is a characteristic matrix for S. It is now easy to see that every infinite boolean algebra, supplemented by the function * as above, is a characteristic matrix for $S5$. It is also proved that every quasi-normal extension of $S5$ is normal, that is, satisfies the rule that if α is provable then $\sim\diamond\sim\alpha$ is provable. *A. Heyting.*

Halldén, Sören. On the semantic non-completeness of certain Lewis calculi. J. Symbolic Logic 16, 127–129 (1951).

Neither (i) $=(\diamond(q.\sim q)\dashv .q.\sim q)$ nor (ii) $=\sim(\diamond(p.\sim p)\dashv .p.\sim p)$ is a theorem of $S3$, but (iii) $=$ (i) $\vee$ (ii) is a theorem of $S1$. Suppose we have an interpretation i of $S3$, in which $\vee$ is interpretated in the usual way; if all $S3$-theorems are true in i, (iii) is true, so either (i) or (ii) is true in i. It follows that the class of $S3$-theorems cannot coincide with the class of true formulas in i. If (i) is added to $S3$ as a new postulate, the result is $S4$; if (ii) is added to $S3$, the result is $S7$ ($S3$ with $\diamond\diamond q$ added). *A. Heyting* (Amsterdam).

Myhill, John. A system which can define its own truth. Fund. Math. 37, 190–192 (1950).

Für Logikkalküle S, die reichhaltig genug sind (insbesondere eine Negation enthalten), lässt sich nach Tarski analog zum Epimenides-Paradox beweisen, dass S keine Definition der Menge aller wahren Aussagen enthält. Mit Hilfe einer 3-stelligen allgemein-rekursiven Funktion von R. Péter [Math. Ann. 111, 42–60 (1935)], die alle 2-stelligen primitiv-rekursiven Funktionen zu definieren gestattet, konstruiert Verf. einen sehr einfachen Kalkül (ohne Negation), der eine Definition der Menge seiner wahren Aussagen enthält. *P. Lorenzen* (Bonn).

Griss, G. F. C. Logic of negationless intuitionistic mathematics. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 41–49 (1951).

The author sketches a propositional calculus intended to formalize his conception of a negationless intuitionistic logic. [same Proc. 53, 456–463 = Indagationes Math. 12, 108–115 (1950); C. R. Acad. Sci. Paris 227, 946–948 (1948); these Rev. 12, 3; 10, 277] The only connectives are implication and conjunction. The principle $A\rightarrow(B\rightarrow A)$ is rejected. Consequently $A\rightarrow(B\rightarrow C)\rightleftarrows A\ \&\ B\rightarrow C$ also fails. Principles such as $A\rightarrow(B\rightarrow C)\rightleftarrows B\rightarrow(A\rightarrow C)$ are also not deducible in the system. The reviewer does not, on the basis of Griss's previously stated views, see any objection to the latter. An algebra of species (sets) is also outlined for the operations of union, complementation, and intersection, and the relation of inclusion. Portions of this work depend on axioms for the "touch condition" introduced to avoid null intersections. Without any indication of a predicate calculus the author gives a list of formulas intended as formal theorems of an as yet unspecified formal system of arithmetic. It would appear that the author intends, in an application of the predicate calculus, that every well-formed part of a true formula be true. Thus the, here unmentioned, morphological rules (definitions of term and formula) for a formal system such as the projected one for arithmetic will be the more complicated portion of the formal equipment. *D. Nelson* (Washington, D. C.).

Markov, A. On an unsolvable problem concerning matrices. Doklady Akad. Nauk SSSR (N.S.) 78, 1089–1092 (1951). (Russian)

The following theorem is proved. If $n\geqq 6$, a set of 102 n-rowed square matrices U_i with rational integral elements can be found, such that the problem of deciding whether any given integral matrix U of the same size is expressible as a finite product $\prod U_{i_r}$ is unsolvable (in the usual sense, that no algorithm or machine routine exists). The proof, which is given in full, depends on an earlier unsolvability theorem of the author on matrices [same Doklady 57, 539–542 (1947); these Rev. 9, 221]. The transformation of the one problem into the other depends on ingenious but elementary algebra. The author believes that the number 102 can probably be lowered. *M. H. A. Newman* (Manchester).

Robinson, Raphael M. Arithmetical definability of field elements. J. Symbolic Logic 16, 125–126 (1951).

Ein Element α eines Körpers K heisst "arithmetisch-definierbar", wenn es eine Aussage $A(x)$, zusammengesetzt aus den Körperoperationen und den logischen Operationen (Variabilitätsbereich der Variablen sei K), gibt mit $x=\alpha\leftrightarrow A(x)$. Jedes arithmetisch-definierbare Element ist fix bezüglich aller Automorphismen von K. Falls K ein algebraischer Zahlkörper ist, beweist Verf. auf eine einfache Weise, dass auch umgekehrt jedes Element α von K, das fix bezüglich aller Automorphismen von K ist, arithmetisch-definierbar ist—und zwar mit Hilfe von Polynomen f, g mit rationalen Koeffizienten durch $x=\alpha\leftrightarrow \bigvee_y(f(y)=0 \wedge x=g(y))$. *P. Lorenzen* (Bonn).

Bouligand, Georges. La nature des choses en mathématiques. Rev. Gén. Sci. Pures Appl. N.S. 58, 131–146 (1951).

The author illustrates insight in mathematics.

C. C. Torrance (Annapolis, Md.).

Aleksandrov, A. D. On certain questions of scientific work and the education of mathematicians. Vestnik Leningrad. Univ. 1950, no. 1, 3–20 (1950). (Russian)

Aleksandrov, A. D. The dialectics of Lenin and mathematics. Vestnik Leningrad. Univ. 1950, no. 4, 24–30 (1950). (Russian)

ALGEBRA

Ryser, H. J. A combinatorial theorem with an application to latin rectangles. Proc. Amer. Math. Soc. 2, 550–552 (1951).

The combinatorial theorem is to the following effect: a square ($n\times n$) matrix composed entirely of zeros and ones and with k ones in each row and column can be obtained from a rectangular ($r\times n$) submatrix if the latter has k ones in each row and the number of ones in any column is not less than $k-n+r$ nor greater than k. This is used to extend Marshall Hall's theorem [Bull. Amer. Math. Soc. 51, 387–388 (1945); these Rev. 7, 106] on the extendibility of a Latin rectangle to a Latin square. It is shown that an $r\times s$ rectangle based on integers 1 to n may be extended to an $n\times n$ Latin square if and only if $N(i)$, the number of appearances of i in the rectangle, is not less than $r+s-n$ for every i.

J. Riordan (New York, N. Y.).

Nair, K. R. Rectangular lattices and partially balanced incomplete block designs. Biometrics 7, 145–154 (1951).

The author shows that the simple rectangular lattices introduced by Harshbarger [Virginia Agricultural Experiment Station, Memoir 1 (1947); these Rev. 10, 202] are partially balanced incomplete block designs whilst the triple rectangular lattices of Harshbarger [Biometrics 5, 1–13 (1949); these Rev. 11, 3] do not necessarily belong to this class of designs. Several examples are discussed in detail.

H. B. Mann (Columbus, Ohio).

Bose, R. C. Partially balanced incomplete block designs with two associate classes involving only two replications. Calcutta Statist. Assoc. Bull. 3, 120–125 (1951).

A partially balanced incomplete block design is termed connected if any two varieties v_1, v_2 can be connected by a sequence of blocks $B_1\cdots B_k$ such that $v_1\subset B_1$, $v_2\subset B_k$, and B_i has at least one variety in common with B_{i+1} ($i<l$). The author shows that the connected partially balanced incomplete block designs with exactly two classes of associates and two replications fall into three series whose parameters are explicitly given in the paper.

H. B. Mann.

Jacobsthal, Ernst. Über eine Determinante. Norske Vid. Selsk. Forh., Trondheim 23, 127–129 (1951).

The note is concerned with the evaluation of the determinant $D_n(x)=\det|a_{ij}(x)|$ $(i, j=1, 2, \cdots, n-1)$, $n\geqq 2$, where $a_{ij}(x)=\{(i-j+1)n+(j-1)\}\binom{x}{i-j+2}$, the symbol $\binom{x}{k}$ being defined as $x(x-1)\cdots(x-k+1)/k!$ when k is a positive integer and as zero in all other cases. This determinant has a bearing on the theory of automorphic functions. With the help of one of his results on the inversion of power series [same Forh. 21, 13–17 (1948); these Rev. 12, 169] the author establishes the formula

$$D_n(x)=x^{n-1}(x-1)(2x-1)\cdots((n-1)x-1).$$

W. Ledermann (Manchester).

Ore, Oystein. Some studies on cyclic determinants. Duke Math. J. 18, 343–354 (1951).

It is well known that the cyclic determinant

$$(1)\quad \mathfrak{C}(a_0, a_1, \cdots, a_{n-1})=\prod_{i=1}^{n}(a_0+a_1\alpha_i+\cdots+a_{n-1}\alpha_i^{n-1})$$

where α_i run through the nth roots of unity. The author obtains, in this paper, an explicit expression for the coefficient of

$$(2)\quad a_0^{\nu_0}a_1^{\nu_1}\cdots a_{n-1}^{\nu_{n-1}},\quad \text{where}\quad \nu_0+\nu_1+\cdots+\nu_{n-1}=n,$$

in the expansion of (1). It is shown that the coefficient vanishes unless the weight P of (2) given by the relation (3) $P=\nu_1+2\nu_2+3\nu_3+\cdots+(n-1)\nu_{n-1}$ is divisible by n. This condition restricts the number and the form of the terms in the expansion. Some interesting relations between the coefficients are established; e.g., it is shown that the sum of the coefficients of the terms for which $P=jn$ is (4) $(-1)^j\binom{n-1}{j}$. Some congruence properties of the coefficients are also discussed for prime values of n. There are two obvious misprints in (19) on p. 347.

H. Gupta (Hoshiarpur).

Schoenberg, I. J., and Whitney, Anne. A theorem on polygons in n dimensions with applications to variation-diminishing and cyclic variation-diminishing linear transformations. Compositio Math. 9, 141–160 (1951).

The authors consider real linear transformations $(y)=A(x)$ where $(x)=(x_1, \cdots, x_n)$, $(y)=(y_1, \cdots, y_m)$ and $A=(a_{ji})$ is of rank r. Let $v(x)$ denote the number of variations of sign in the sequence $x_1, \cdots, x_n$ and similarly for $v(y)$. It is shown that $\sup v(y)\geqq r-1$. On the other hand, $v(y)\leqq r-1$ for all choices of x if and only if the matrix $A^{(r)}$ has definite columns, that is, each of its columns is a definite one-column matrix, a real matrix being called definite if it has no two elements of opposite signs. Here $A^{(i)}$, $1\leqq i\leqq r$, is the matrix of $\binom{m}{i}$ rows and $\binom{n}{i}$ columns, whose elements are the ith order minors of A where all minors formed from the same set of i rows of A appear in the same row of $A^{(i)}$ and the same rule holds for the columns. The condition $v(y)\leqq r-1$ is given a geometrical interpretation where $v(y)$ represents the number of times a certain hyperplane in E_n is crossed by the polygonal line $P_1P_2\cdots P_n$, $P_i=(a_{i1}, \cdots, a_{in})$. As an application a proof is given of the theorem of Motzkin [Basel dissertation (1933), Jerusalem, 1936] according to which $v(y)\leqq v(x)$, that is the transformation is variation-diminishing, if and only if the matrices $A, A^{(2)}, \cdots, A^{(r-1)}$ are definite and $A^{(r)}$ has definite columns. If cyclic variations are considered instead, the authors give necessary and sufficient conditions for $v_c(y)\leqq v_c(x)$. Here the matrices $A^{(i)}$ of odd order are definite and, if r is odd, $A^{(r)}$ has definite columns, while if r is even a more complicated condition enters involving sign changes of cyclic minors of order r.

E. Hille (New Haven, Conn.).

Parodi, Maurice. Compléments à un travail sur la stabilité. J. Phys. Radium (8) 12, 665–666 (1951).

Conditions are given sufficient to ensure that the determinantal equation $|p_{ik}(z)|=0$ has only roots with non-positive real parts, where $p_{ii}=(a_{11}+b_{11}z)(\alpha_{11}+\beta_{11}z)^{-1}$ and $p_{ik}=a_{ik}$ for $i\neq k$. [See also M. Parodi, same J. (8) 10, 200–201 (1949); these Rev. 10, 671.] *O. Taussky-Todd.*

Holmyanskiĭ, M. M. On the solution of systems of algebraic equations of the fundamental problems of the plane theory of elasticity and of some problems of the engineering theory of the bending of thin plates. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 317–322 (1951). (Russian)

The note is concerned with the solution of the system of linear algebraic equations

$$a_{jj}x_j+\sum_{k=1,\,k\neq j}^{u-j+1} a_{jk}x_k=b_j,\qquad j=1,2,\cdots,u,$$

arising in certain problems indicated in the title of the paper. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Remez, E. Ya. On Čebyšev approximations in a complex region. Doklady Akad. Nauk SSSR (N.S.) 77, 965–968 (1951). (Russian)

The author studies the problem of a "best" approximate solution of a system $\sum_1^n a_i^{(\alpha)}z_i-l^{(\alpha)}=0$ in the complex domain. The measure of the approximation is the maximum with respect to α of the absolute value of the left hand sides. The solutions of the problem form a convex set in a complex Euclidean n-space. He studies the problem for subsystems of the above and proves several theorems. *František Wolf* (Berkeley, Calif.).

Bell, James H. A note on the solution of the unilateral matrix equation. Proc. Amer. Math. Soc. 2, 553–557 (1951).

An algorithm of M. H. Ingraham [Bull. Amer. Math. Soc. 47, 61–70 (1941); these Rev. 2, 243] for the solution of the unilateral matrix equation $\sum_{i=0}^{r}R_iX^i=0$ is here extended by allowing the R_i to be m by n matrices instead of n by n. If $m<n$ and there is one solution, an infinite number exist. *W. Givens* (Knoxville, Tenn.).

Morinaga, Kakutarô, and Nôno, Takayuki. On the logarithmic functions of matrices. I. J. Sci. Hiroshima Univ. Ser. A. 14, 107–114 (1950).

Let M be a regular matrix with complex coefficients. The authors discuss the solutions in matrices A of the equation $\exp A=M$. They first prove that there always exists a unique solution $A=L(M)$ which satisfies the following condition: the imaginary parts of the characteristic roots of A lie in the half-closed interval $[-\pi,+\pi)$. Moreover, the mapping $M\rightarrow L(M)$ induces a homomorphism on the set of regular matrices M which do not have any real negative characteristic root. Next, the general solution of the equation is determined. The result may be expressed as follows. Considering M as an automorphism of a vector space V, let $V=V_1+\cdots+V_h$ be a direct sum decomposition of V into spaces stable for M and indecomposable relatively to this property. Then the general solution is $A=L(M)+S^{-1}FS$ where S commutes with M and F is an endomorphism which transforms each V_i into itself and whose restriction to V_i is of the form $2\pi(-1)^{\frac{1}{2}}f_iE_i$, f_i being an integer and E_i the identity mapping of V_i. Among all these solutions, the authors determine those which may be written as polynomials in M; they are characterized by the condition that $f_i=f_j$ whenever V_i and V_j belong to the same characteristic root of M. Finally, F being given, they determine the number of essential parameters on which the solution $L(M)+S^{-1}FS$ depends. *C. Chevalley.*

Morinaga, Kakutarô, and Nôno, Takayuki. On the logarithmic functions of matrices. II. (On some properties of local Lie groups). J. Sci. Hiroshima Univ. Ser. A. 14, 171–179 (1950).

The first part of the paper is concerned with the extension of the results of the preceding paper to the case of real matrices M. Using the notation of the preceding review, it is proved that, if M is real, then the real and imaginary parts of $L(M)$ commute with each other. If M has no real negative characteristic root, then $L(M)$ is itself real.

The second part of the paper contains applications to the theory of Lie groups. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra; then, if $X\in\mathfrak{g}$, the adjoint representation of G represents $\exp X$ by $\exp(\operatorname{ad} X)$, where $\operatorname{ad} X$ is the image of X in the adjoint representation of $\mathfrak{g}$. Since $\operatorname{ad} X$ is a solution of the equation $\exp A=\operatorname{ad}(\exp X)$, it is possible in certain cases to infer properties of $\operatorname{ad} X$ from certain properties of the single element $\exp X$; and these properties of $\operatorname{ad} X$ will give properties of all elements $\exp tX$ of the one parameter group generated by X. A typical result of this kind is the following. Denote by A^* the set of endomorphisms of $\mathfrak{g}$ no two characteristic roots of which differ from each other by an integral multiple of $2\pi(-1)^{\frac{1}{2}}$. Let X and Y be elements of $\mathfrak{g}$ such that $\exp X$ and $\exp Y$ commute with each other; if $\operatorname{ad} X$ and $\operatorname{ad} Y$ belong to A^*, then X and Y commute with each other, from which it follows that $\exp tX$ commutes with $\exp uY$ for every real t and u. *C. Chevalley.*

★Gel'fand, I. M. Lekcii po lineĭnoĭ algebre. [**Lectures on Linear Algebra**]. 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 252 pp.

This book covers the standard topics in linear algebra. The most noteworthy feature is the insertion of two appendices on computational methods. The headings are as follows. I. n-dimensional spaces. Linear and bilinear forms. II. Linear transformations. III. Canonical form of a linear transformation. IV. The concept of tensor. Appendix I. Methods of computation in linear algebra. II. The theory of perturbation. *I. Kaplansky* (Chicago, Ill.).

Abstract Algebra

Fraïssé, Roland. Sur la signification d'une hypothèse de la théorie des relations, du point de vue du calcul logique. C. R. Acad. Sci. Paris 232, 1793–1795 (1951).

An unsolved problem in the author's theory of multi-relations [same C. R. 230, 1557–1559 (1950); these Rev. 12, 14] is here reformulated as a statement concerning the equivalence of pairs of formulas in the predicate calculus. *R. Arens* (Los Angeles, Calif.).

Wendelin, Hermann. Untersuchungen zur Mengenalgebra. J. Reine Angew. Math. 188, 78–99 (1950).

Wendelin, Hermann. Ein Vergleichskriterium für Ausdrücke in Booleschen Verbänden und einige Anwendungen. J. Reine Angew. Math. 188, 147–149 (1950).

The well-known process of reducing a Boolean function to a normal form is used to solve equations over a set-field. *B. Jónsson* (Providence, R. I.).

Petropavlovskaya, R. V. Structural isomorphisms of free associative systems. Mat. Sbornik N.S. 28(70), 589–602 (1951). (Russian)

The subsystems of an associative system A form a lattice L. If two systems A_1 and A_2 are isomorphic or anti-isomorphic, then the lattices L_1 and L_2 are isomorphic, but the converse isomorphism of L_1 and L_2, called a structural isomorphism of A_1 and A_2 is a weaker relationship. It is shown that an arbitrary system A can be embedded into two structurally isomorphic systems A_1 and A_2 which are neither isomorphic nor anti-isomorphic, and in such a way that the structural isomorphism between A_1 and A_2 carries every subsystem of A into itself. This is shown by a simple construction. The main part of the paper is then devoted to showing that an associative system B structurally isomorphic to a free system A is also free, and the correspondence between B and A is an isomorphism or an anti-isomorphism. *Marshall Hall* (Washington, D. C.).

Livšic, A. H. Direct decompositions of complete Dedekind structures. Mat. Sbornik N.S. 28(70), 481–502 (1951). (Russian)

This paper is a continuation of the program inaugurated by Kuroš [Izvestiya Akad. Nauk SSSR 7, 185–202 (1943); 10, 47–72 (1946); these Rev. 6, 145; 8, 309] of studying theorems of the Krull-Schmidt type within the framework of lattice theory. The main objective is to generalize to lattices the theorems proved for loops by Baer [Trans. Amer. Math. Soc. 62, 62–98 (1947); these Rev. 9, 134]. One assumes a complete modular lattice S. Let θ_1, θ_2 be the complementary endomorphisms induced by a direct decomposition of the unit of S; let ϕ be a product of such endomorphisms. Then $\phi\theta_1\phi\theta_2\phi$ is called a distinguished endomorphism. S is said to be regular if every distinguished endomorphism satisfies two conditions too elaborate to reproduce here. These conditions at any rate follow from the continuity assumption $x\cap(\sum y_i)=\sum(x\cap y_i)$; this latter assumption is made by Grayev [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 11, 33–46 (1947); these Rev. 8, 560] and Hostinsky [Duke Math. J. 18, 331–342 (1951); these Rev. 12, 795], for similar purposes. Finally, S is said to satisfy the splitting hypothesis if for every distinguished endomorphism there is a complement in the usual way. The main theorem asserts that regularity of S, plus the splitting hypothesis, imply that any two decompositions of S into two summands have isomorphic refinements in the sense of Baer [loc. cit., p. 77]. From this point, the passage to any (finite) number of summands is taken from Baer, since this part of Baer's work uses only lattice-theoretic arguments. The paper concludes with theorems showing how regularity and the splitting hypothesis can be deduced from various weak chain conditions.

I. Kaplansky (Chicago, Ill.).

Takeuchi, Kensuke. On maximal proper sublattices. J. Math. Soc. Japan 2, 228–230 (1951).

An elementary distributive lattice is exhibited, containing a chain (hence sublattice) which cannot be extended to a maximal proper sublattice. It is shown that, in a Boolean algebra, every proper sublattice can be extended to a maximal proper sublattice. These results solve Problem 18 of the reviewer's "Lattice Theory", 2d ed. [Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948; these Rev. 10, 673].

G. Birkhoff (Cambridge, Mass.).

Green, J. A. On the structure of semigroups. Ann. of Math. (2) 54, 163–172 (1951).

L'auteur suit ici la terminologie déjà utilisée par D. Rees [Proc. Cambridge Philos. Soc. 36, 387–400 (1940); 37, 434–435 (1941); ces Rev. 2, 127; 3, 199] en appelant semigroupe (=demigroupe au sens de P. Dubreil=monoide au sens de N. Bourbaki) un ensemble S muni d'une loi partout définie associative. Un idéal à gauche de S est un sous-ensemble A tel que $SA\subset A$. Si on désigne par xLy (resp. xRy, resp. xFy) la relation d'équivalence "x et y engendrent le même idéal principal à gauche" (resp. à droite, resp. bilatère) on a $RL=LR\subset F$. Si on désigne respectivement par $\mathfrak{M}_l$, $\mathfrak{M}_r$, $\mathfrak{M}_f$, $\mathfrak{M}_s$ l'hypothèse de la condition minimale sur l'ensemble ordonné par inclusion des idéaux principaux à gauche, resp. à droite, resp. bilatère, resp. des sous-semi-groupes principaux de S on démontre les propositions suivantes: Si tout élément de S est d'ordre fini (ce qui revient à dire que S satisfait à $\mathfrak{M}_s$) on a $RL=LR=F$; dans toute classe mod RL, toute classe mod L est minimale et tout élément simple a de S (c'est-à-dire tel que $a\epsilon F(a)\cdot a\cdot F(a)$, $F(a)$ étant la classe des éléments congrus à a mod F) est régulier (c'est-à-dire tel que $a\epsilon aSa$). Si S satisfait à $\mathfrak{M}_l$ et à $\mathfrak{M}_r$, S satisfait à $\mathfrak{M}_f$ et ce qui précède est encore vrai. Les classes mod F de S correspondent aux "facteurs principaux" de D. Rees. S est dit semisimple si tous ces facteurs sont des semigroupes non nilpotents. Lorsque S satisfait à $\mathfrak{M}_l$ et $\mathfrak{M}_r$, le concept de semisimplicité coincide avec celui de régularité (S étant dit régulier si tous ses éléments le sont). En utilisant ces résultats, on peut déduire la structure d'un semigroupe complètement simple au sens de D. Rees. *J. Riguet* (Paris).

Clark, John F., Jr. Rings associated with the rings of endomorphisms of finite groups. J. Washington Acad. Sci. 40, 385–397 (1950).

For an additively written group G one defines in a well known manner addition and multiplication in the set $T_{(G)}$ of all single-valued transformations of G onto subsets of G. Then all postulates of a ring are valid in $T_{(G)}$ except $a+b=b+a$ and $(b+c)a=ba+ca$. The subset $E_{(G)}$ of $T_{(G)}$ of all endomorphisms of G forms a ring in case G is commutative, but in the general case is not closed under addition, which also need not be commutative. The author discusses for some special finite groups the problem as to the occurrence of rings among the subsets of $T_{(G)}$ and $E_{(G)}$. The general (finite) cyclic group and the general dihedral group are treated, and the groups of order $\leqq 6$ are discussed in detail.

J. Levitski (Jerusalem).

√ ***Apéry, R. Quelques propriétés des anneaux.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 107–108. Centre National de la Recherche Scientifique, Paris, 1950.

A heterogeneous ring is a set A of elements partitioned into classes by an equivalence relation and possessing two operations. Addition is possible only within each equivalence class. Under addition each class forms an abelian group. Each class has an additive identity which is a zero of the ring. Multiplication is defined throughout and is commutative, associative, and distributive with respect to addition. Multiplication is a well-defined operation with respect to the equivalence relation, and the equivalence classes con-

stitute an abelian group with respect to this operation. An element is called regular if it is not a divisor of one of the zeros. An ideal is a subset which includes the sum of any two of its elements which are equivalent and contains the product of any element of the subset with any element of the ring. An ideal is regular if it contains a regular element. The terms union of ideals, principal ideal, etc. are defined as usual.

The ideal $\mathfrak{b}$ is independent of the ideal $\mathfrak{a}$ if $\mathfrak{a}:\mathfrak{b}=\mathfrak{a}$. The ideal $\mathfrak{b}$ is algebraically complete if $\mathfrak{b}\cdot\mathfrak{a}:\mathfrak{a}=\mathfrak{b}$ for every regular ideal $\mathfrak{a}$. A regular ideal $\mathfrak{a}$ is called immediate if $\mathfrak{a}=\{(\varphi):[(\varphi):\mathfrak{a}]\}$ for some regular element φ contained in $\mathfrak{a}$. The associate of a regular ideal $\mathfrak{a}$ is the ideal $\mathfrak{a}'=(\varphi):\{\varphi\cdot\mathfrak{a}:\mathfrak{a}\}$ where φ is a regular element of $\mathfrak{a}$. An immediate ideal $\mathfrak{a}$ is normal if its associate is independent of $\mathfrak{a}$, otherwise it is anormal. An immediate ideal is absolutely normal if its associate is the ring A.

A typical theorem (stated without proof) concerning the properties of these ideals is the following: A necessary and sufficient condition that every regular principal ideal be algebraically complete is that every immediate ideal shall be absolutely normal. As an example of a heterogeneous ring, let A consist of homogeneous polynomials in $n>1$ variables with coefficients in a field. Let the equivalence classes consist of polynomials of the same degree with the exception of classes of polynomials of negative degree in which all polynomials with non-zero coefficients are excluded. The group of classes is isomorphic with the additive group of the rational integers. It is remarked that if the ring arises from a variety of dimension d, all singular varieties of dimension $d-1$ and certain singular varieties of lower dimension give rise to immediate anormal ideals.

F. Kiokemeister (Chicago, Ill.).

Amitsur, S. A. Nil *PI*-rings. Proc. Amer. Math. Soc. 2, 538–540 (1951).

The author sharpens previous results of Levitzki [same Proc. 1, 334–341 (1950); these Rev. 12, 6]. Let S be a ring satisfying a polynomial identity of degree d, $m=[d/2]$, T a nil subring of S, and N the union of the nilpotent ideals of S; then $T^m\subset N$. It follows as a corollary that if S is a nil ring, the radical series from 0 to S has length at most 2.

I. Kaplansky (Chicago, Ill.).

Mori, Shinjiro. Über die Symmetrie des Prädikates "relativ prim." J. Sci. Hiroshima Univ. Ser. A. 14, 102–106 (1950).

In a commutative ring R the ideal $\mathfrak{b}$ is said to be prime relative to the ideal $\mathfrak{a}$ if $\mathfrak{a}=\mathfrak{a}:\mathfrak{b}$. This relation is said to be symmetric in R if, for non-zero ideals, $\mathfrak{a}=\mathfrak{a}:\mathfrak{b}$ always implies $\mathfrak{b}=\mathfrak{b}:\mathfrak{a}$. The author characterizes certain rings which have the property of this symmetry, assuming in each case that the ring contains proper ideals. It is shown, for example, that the relation "prime relative to" is symmetric in the commutative ring R if and only if for each divisor $\mathfrak{b}$, $\mathfrak{b}\neq R$, of an ideal $\mathfrak{a}$, $\mathfrak{a}\neq R$, $\mathfrak{a}\neq(0)$, there is an element r non-$\epsilon\,\mathfrak{a}$ such that $(r)\mathfrak{b}\subseteq\mathfrak{a}$, and, if r non-$\epsilon\,\mathfrak{a}$, $\mathfrak{a}\neq(0)$, then $(r)R\not\subseteq\mathfrak{a}$. If every ideal distinct from R is contained in a maximal ideal and if the relation "prime relative to" is symmetric in R, then every ideal distinct from (0) is the intersection of finitely or infinitely many primary ideals belonging to maximal ideals. Theorem 6 reads as follows: Let R be a commutative ring with ascending chain condition. Then the relation "prime relative to" is symmetric in R if and only if (1) R has a unit, and (2) the descending chain condition holds in R. The ring of rational integers constitutes a counter-example to the theorem.

F. Kiokemeister (Chicago, Ill.).

Mori, Shinjiro. Teilerfremde und relativ prime Ideale. J. Sci. Hiroshima Univ. Ser. A. 14, 165–169 (1950).

The author poses the following problem: What is the character of a commutative ring R in which, for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$, $\mathfrak{a}=\mathfrak{a}:\mathfrak{b}$ or $\mathfrak{b}=\mathfrak{b}:\mathfrak{a}$ implies $(\mathfrak{a},\mathfrak{b})=R$, and conversely $(\mathfrak{a},\mathfrak{b})=R$ implies $\mathfrak{a}=\mathfrak{a}:\mathfrak{b}$ or $\mathfrak{b}=\mathfrak{b}:\mathfrak{a}$; i.e., in which the relations "relatively prime" and "prime relative to" for ideals are equivalent. In a commutative ring with unit element, the two relations are equivalent if and only if the relation "prime relative to" is symmetric [see the preceding review]. If R satisfies the ascending chain condition for ideals, then the two relations are equivalent if and only if every non-zero prime ideal in R is maximal and there is no ideal between R and R^2.

F. Kiokemeister (Chicago, Ill.).

✓***Lesieur, L. Le transfert de certaines propriétés d'un anneau *A* à l'anneau des polynomes *A*[*x*].** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 99–101. Centre National de la Recherche Scientifique, Paris, 1950.

A summary of some properties of a ring A that are inherited by the polynomial ring $A[x]$. Proof of Hilbert's Nullstellensatz as in a paper of the author [Canadian J. Math. 2, 50–65 (1950); these Rev. 11, 310].

D. Zelinsky (Evanston, Ill.).

Azumaya, Gorô. New foundation of the theory of simple rings. Proc. Japan Acad. 22, no. 11, 325–332 (1946).

Nakayama, Tadasi. Note on irreducible rings. Proc. Japan Acad. 22, no. 11, 333–337 (1946).

The essential results of these papers are contained in a paper by T. Nakayama and G. Azumaya [Ann. of Math. (2) 48, 949–965 (1947); these Rev. 9, 563]. *R. Brauer.*

Kaplansky, Irving. A theorem on division rings. Canadian J. Math. 3, 290–292 (1951).

The author proves the following. Let A be a division ring with center Z, and suppose that for every x in A, some power is in Z. Then A is commutative. This contains the Wedderburn-Jacobson and Noether-Jacobson theorems as well as a theorem of the reviewer as special cases.

L.-K. Hua (Peking).

***Châtelet, A. Idéaux principaux dans les corps circulaires.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 103–106. Centre National de la Recherche Scientifique, Paris, 1950.

Let C be a cyclotomic field generated by a primitive dth root of unity, where d is odd or a multiple of 4. Let A be a principal ideal (integral or fractional) in C. Let $A_1=A$, A_2, $\cdots$ be the conjugates of A in C. If, for each A_i, $A_1{}^i\times A_i{}^{-1}=B_{1i}{}^d$ where B_{1i} is principal, then A is called an essential ideal. This definition is invariant under the automorphisms of C. If P is a prime ideal of prime degree with conjugates $P_1=P, P_2, \cdots$, then $Q=\prod P_i{}^j$, where $i\cdot j\equiv a \pmod d$ and $1\leqq j<d$, is an essential ideal for each integer a (mod d). An ideal A is principal if and only if A^d is essential. The role of essential ideals in the construction of abelian fields is explained. *F. Kiokemeister.*

Jaffard, Paul. Sur certains types de modules. C. R. Acad. Sci. Paris 233, 13–15 (1951).

Let A be a ring, M a right A-module, K a field, $E=E/K$ a right vector space over K. Then M is called equivalent to E if there exists a homomorphism φ of A onto K and a one-one mapping f of M onto E such that $f(x-y)=f(x)-f(y)$ and $f(xa)=f(x)\varphi(a)$ for all $x\epsilon M$, $y\epsilon M$, $a\epsilon A$. The principal result is the following: A necessary and sufficient condition that $M\neq(0)$ be equivalent to some E/K is: (1) $MA\neq(0)$; (2) xA is irreducible for all $x\epsilon M$; and (3) if M is irreducible then the annihilator of each $x\epsilon M$ is equal to the annihilator B of M. In this case, $A-B$ is a field and K can be taken to be $A-B$. The proof is based on a straightforward analysis of direct sums of irreducible A-modules. *G. K. Kalisch.*

Steinfeld, Ottó. Bemerkung zu einer Arbeit von L. Kalmár. Publ. Math. Debrecen 2, 48–49 (1951).

Let K be a subfield of the field of real numbers, B the ring of bounded sequences of elements of K, C the subring of B consisting of Cauchy sequences, N the ideal in B consisting of sequences tending to 0. The author proves the following conjecture of L. Kalmár: for every ring D such that $C\subset D\subset B$ and $C\neq D$, N is not a maximal ideal in D.

J. Dieudonné (Nancy).

✓ ***Artin, E. Remarques concernant la théorie de Galois.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 161–162. Centre National de la Recherche Scientifique, Paris, 1950.

The paper contains a number of remarks concerning proofs of theorems of Galois theory. The following results may be mentioned. Let E be an extension field of finite degree n over a given field F, let σ be an isomorphism of F into a field Ω. The field Ω is said to be an absorbing field for σ, if every extension of σ to an isomorphism of E into an extension field of Ω actually maps E into Ω. In this case, there exist exactly m extensions $\sigma_1, \cdots, \sigma_m$ of σ to isomorphisms of E into Ω where m is the separable part of the degree of E over F. Let $\omega_1, \omega_2, \cdots, \omega_n$ denote a basis of E over F, let $\xi=Y_1\omega_1+\cdots+Y_n\omega_n$ denote the generic element of E; $Y_i\epsilon F$; $\sigma_i(\xi)=X_1\sigma_i(\omega_1)+\cdots+X_n\sigma_i(\omega_n)$ where $X_j=\sigma(Y_j)$. A function $\Phi(\xi)$ of ξ with values in Ω is said to be semi-analytic, if it can be expressed as a polynomial in $X_1, \cdots, X_n$ with coefficients in Ω. The function $\Phi(\xi)$ is said to be analytic, if it can be expressed as a polynomial in $\sigma_1(\xi), \cdots, \sigma_m(\xi)$ with coefficients in Ω. A function $\Phi(\xi)$ is analytic, if and only if (1) $\Phi(\xi)$ is semi-analytic, and (2) an equation $[\Phi(\xi)]^{n/m}=g(\xi^{n/m})$ holds where $g(\xi)$ is a semi-analytic function. This yields the following characterization of the norm: The norm is a multiplicative homomorphism of E into F which can be given by a semi-analytic function of degree n.

R. Brauer (Ann Arbor, Mich.).

Samuel, Pierre. Corps valués quasi algébriquement clos. C. R. Acad. Sci. Paris 232, 1985–1987 (1951).

A field K is said to be quasi algebraically closed (q.a.c.) if every homogeneous equation of degree d in s unknowns with coefficients in K and with $d<s$ has a non-trivial solution in K. In this note it is shown that if k is an algebraically closed field, the fields $k(T)$ and $k\{\{T\}\}$ of rational functions and formal power series in one variable over k are q.a.c. Also, the field of power series with complex coefficients which converge in the neighborhood of the origin is q.a.c. There is a partial converse to these results: the residue class field of a q.a.c. field with a discrete valuation is algebraically closed.

B. N. Moyls (Vancouver, B. C.).

Braconnier, J. Les nombres p-adiques et quelques-unes de leurs applications. Cahiers Rhodaniens 2, 9 pp. (1950).

This is an expository article on the structure of the p-adic field Q_p and sketches briefly some peculiar function-theoretic possibilities. The elementary functions can all be defined. There are continuous functions with values in Q_p which are not differentiable; continuous functions have primitives, but because of the total disconnection, there are functions constant on no open set whose derivatives vanish. The remarks on complex-valued functions, in particular on harmonic analysis, are just those following from the fact that Q_p is a locally compact group. *R. Arens* (Los Angeles, Calif.).

Kuroš, A. G. The present status of the theory of rings and algebras. Uspehi Matem. Nauk (N.S.) 6, no. 2(42), 3–15 (1951). (Russian)

This is an expository account, with no proofs. A bibliography of 32 recent papers is included. Much attention is devoted to non-associative rings (free, alternative, Jordan, Lie). *I. Kaplansky* (Chicago, Ill.).

Campbell, H. E. An extension of the "principal theorem" of Wedderburn. Proc. Amer. Math. Soc. 2, 581–585 (1951).

Let ξ be the general element of an alternative algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ and $f(\lambda, \xi)=\lambda^m-T(\xi)\lambda^{m-1}+\cdots+(-1)^mT_m(\xi)$ be the minimum function of ξ. Then $T(x)$ has the usual linearity, commutativity and associativity properties of trace functions, and defines an ideal of $\mathfrak{A}$, its "first liberal". This is the set $\mathfrak{T}_1$ of all x in $\mathfrak{A}$ such that $T(xy)=0$ for every y of $\mathfrak{A}$. When $\mathfrak{A}$ is semisimple $\mathfrak{T}_1$ is the direct sum of the inseparable components of $\mathfrak{A}$. There exists an ideal $\mathfrak{T}$ of $\mathfrak{A}$ such that $\mathfrak{T}-\mathfrak{N}$ is the first liberal of $\mathfrak{A}-\mathfrak{N}$, where $\mathfrak{N}$ is the radical of $\mathfrak{A}$. This unique ideal is called the "liberal" of $\mathfrak{A}$. The author then shows that if $\mathfrak{T}$ is the liberal of an alternative algebra $\mathfrak{A}$ there exists an algebra $\mathfrak{S}$ such that $\mathfrak{A}=\mathfrak{S}+\mathfrak{T}$.

A. A. Albert (Chicago, Ill.).

Jacobson, N. Enveloping algebras of semi-simple Lie algebras. Canadian J. Math. 2, 257–266 (1950).

Let $\mathfrak{L}$ be a Lie algebra over a field Φ, let $\mathfrak{A}$ be an associative algebra over Φ and let $\mathfrak{A}_l$ be the Lie algebra associated with $\mathfrak{A}$. An imbedding of $\mathfrak{L}$ in $\mathfrak{A}$ is a homomorphism $S: a\to a^S$ of $\mathfrak{L}$ into $\mathfrak{A}_l$; the subalgebra of $\mathfrak{A}$ generated by all elements $a^S\epsilon\mathfrak{A}$ is the enveloping algebra $\mathfrak{E}_S$ of the imbedding S. If S, T are two imbeddings of $\mathfrak{L}$ in $\mathfrak{A}$, $\mathfrak{B}$ with enveloping algebras $\mathfrak{E}_S$, $\mathfrak{E}_T$ respectively, then S "covers" $T(S\geqq T)$ if there is a homomorphism of $\mathfrak{E}_S$ onto $\mathfrak{E}_T$. (In particular, therefore, the familiar Birkhoff-Witt imbedding U covers every imbedding.) A subset Γ of $\mathfrak{L}$ is a "total subset" of $\mathfrak{L}$ if the ideal generated by Γ is $\mathfrak{L}$ itself.

The author obtains the following principal results. Let $\mathfrak{L}$ be a semi-simple Lie algebra of finite dimension over a field of characteristic 0, and Γ a total subset of $\mathfrak{L}$. Then (1) if S is an imbedding of $\mathfrak{L}$ such that, for each $c\epsilon\Gamma$, c^S is algebraic, then $\mathfrak{E}_S$ is of finite dimension; (2) if for fixed t, $\{S\}$ is the set of imbeddings such that, for every $c\epsilon\Gamma$, c^S is algebraic of degree $\leqq t$, then there exists an imbedding T of $\mathfrak{L}$ such that $\mathfrak{E}_T$ is of finite dimension and $T\geqq S$ for all $S\epsilon\{S\}$; (3) there exists only a finite number of inequivalent irreducible representations of $\mathfrak{L}$ such that the degree of the minimum polynomial of every c^S, $c\epsilon\Gamma$, does not exceed a fixed integer. The proofs are made by passing to an algebraically closed extension of Φ and making full use of the structure theory of semi-simple Lie algebras. An application is given to the

problem of determining irreducible inequivalent sets of linear transformations x_i which satisfy the conditions

$$[[x_i x_j]x_k] = \delta_{ki}x_j - \delta_{kj}x_i, \quad i, j, k = 1, 2, \cdots, n,$$
$$\varphi(x) = 0, \quad x = \sum \xi_i x_i \neq 0, \quad \varphi(\lambda) \text{ a polynomial.}$$

A special case of this last was considered earlier by the author [Amer. J. Math. 71, 149–170 (1949); these Rev. 10, 426]. *S. A. Jennings* (Vancouver, B. C.).

Schenkman, Eugene. A theory of subinvariant Lie algebras. Amer. J. Math. 73, 453–474 (1951).

It was proved by Wielandt [Math. Z. 45, 209–244 (1939)] that for a group G with center (1) the tower of automorphism groups $A(G)$, $A(A(G))$, $\cdots$ is finite, i.e. that one eventually reaches a group all of whose automorphisms are inner. This was accomplished by means of a theory of subinvariant subgroups. The present paper establishes an analogous theory of subinvariant subalgebras of a Lie algebra and, in particular, yields a simple proof for the finiteness of the tower of derivation algebras of a Lie algebra with center (0). For base fields of characteristic zero, this result was obtained by Chevalley as a by-product of his theory of algebraic Lie algebras [Proc. Nat. Acad. Sci. U. S. A. 30, 274–275 (1944); these Rev. 6, 201].

A subalgebra A of a Lie algebra L is called subinvariant if it occurs in a normal series for L. An important auxiliary result which has no analogue in the case of groups is that if A is subinvariant in L then $A^\omega = \bigcap_{i=1}^\infty A^i$ is an ideal of L, where $A^1 = A$ and $A^{i+1} = [A^i, A]$. Moreover, $A = A^\omega + H$, where H is a nilpotent subalgebra of L. From this it is deduced that if the centralizer of A in L is (0) then the centralizer of A^ω is contained in A^ω. Now let L be a Lie algebra with center (0). Let $D_0 = L$, and let D_{i+1} denote the derivation algebra of D_i. Then each D_i has center (0) and may be regarded in the natural fashion as an ideal of D_{i+1}. In this sense, L is subinvariant in D_n, and one shows inductively that the centralizer of L in D_n is (0). Application of the last theorem quoted above gives the main result which says that the dimension of D_n is at most equal to the sum of the dimensions of the center of L^ω and the derivation algebra of L^ω. The tower theorem is evidently implied by this.

Apart from these results the paper contains a systematic theory of subinvariant subalgebras, mostly under the assumption that the base field be of characteristic zero. The simplest results are the following: The radical and the maximal nilpotent ideal of a subinvariant subalgebra A of L are the intersections with A of the radical and the maximal nilpotent ideal, respectively, of L. The subalgebra generated by two subinvariant subalgebras is subinvariant. The minimal subinvariant subalgebra of L containing a given element a has the form $I + (a)$, where I is an ideal of L.

G. Hochschild (New Haven, Conn.).

Raffin, R. Axiomatisation des algèbres génétiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 359–366 (1951).

A "genetic domain" D is defined by abstract postulates and extended to a linear algebra A over the real field R. The algebra A can be interpreted as a genetic algebra [Etherington, Proc. Roy. Soc. Edinburgh 59, 242–258 (1939); ibid. Sect. B. 61, 24–42 (1941); these Rev. 1, 99; 2, 237] and D as the subset consisting of all elements $x = \sum \xi_i a_i$ with non-negative coefficients ξ_i, where the basis elements $a_1, \cdots, a_n$ represent genetic types. Using Schafer's characterisation of genetic algebras [Amer. J. Math. 71, 121–135 (1949); these Rev. 10, 350], which is valid when the inheritance referred to is symmetric in the sexes, the author shows that the weight function of A is unique, a weight function being a non-trivial homomorphism of A onto R. Weaker forms of this theorem, and other results, are valid also for the genetic domains and genetic algebras which describe non-symmetric types of inheritance. *I. M. H. Etherington.*

Curry, Haskell B. Note on a theorem on abstract differential equations. Portugaliae Math. 10, 23–24 (1951).

A. S. Amitsur's [Bull. Amer. Math. Soc. 54, 937–941 (1948); these Rev. 10, 231] theorem that a linear differential equation (abstract differentiation over a quasi-field) of order n has at most n linearly independent solutions follows also by elementary methods introduced by Curry [Amer. Math. Monthly 56, 398–402 (1949)]. *G. Whaples.*

Cohn, Richard. Singular manifolds of difference polynomials. Ann. of Math. (2) 53, 445–463 (1951).

The author continues his investigation of the structure of the manifold of a system of difference polynomials over a difference field of characteristic zero [cf. Trans. Amer. Math. Soc. 64, 133–172 (1948); Bull. Amer. Math. Soc. 54, 917–922 (1948); 55, 595–597 (1949); these Rev. 10, 4, 178, 675]. In this paper every essential (i.e. maximal) irreducible component of the manifold of an algebraically irreducible polynomial F in n unknowns $y_1, \cdots, y_n$ is shown to be of dimension $n-1$, in the sense that $n-1$ is the number of arbitrary unknowns in the prime ideal determined by the component. Also, every essential irreducible singular component of F is shown to determine a prime ideal which is of order and effective order in each y_i at least two less than the corresponding order and effective order of F in that y_i. (This last phenomenon has no analogue in the theory of differential polynomials.) The theorem on order and effective order follows from a result, Theorem II of this paper, for difference polynomials, which is analogous to the necessity part of Ritt's low power theorem for differential equations [Ann. of Math. (2) 37, 552–617 (1936)]. The statement of Theorem II being somewhat long, we state here only the following special case, the author's Theorem I, which applies when F has only one unknown y, and vanishes when $y=0$: In order that $y=0$ be an essential singular manifold of F, it is necessary that F contain a single term G such that whenever τ is any positive number, and the weights $1, \tau, \tau^2, \cdots, \tau^p$ are assigned to y and its transforms $y_1, y_2, \cdots, y_p$, respectively, then G is of lower weight than every other term of F. The author proves also that this necessary condition is sufficient in the special case where F has just two terms, both of positive degree. The sufficiency here follows readily from a lemma to the effect that if the polynomial $f(t)$ has real coefficients and has no positive root, then there exists a polynomial $g(t)$, with all coefficients positive or all coefficients negative, such that $f(t)g(t)$ has all coefficients positive. As a corollary of Theorem II the author shows that the essential singular manifolds of F annul all the formal partial derivatives $\partial^k F/(\partial y_{i t_i})^k$, where $y_{i t_i}$ is either the highest transform of y_i appearing in F, or the lowest. The chief method of proof is the construction, when $\sum$ is a prime reflexive ideal held by F, with general point $\alpha_1, \alpha_2, \cdots, \alpha_n$, and when subsidiary conditions are fulfilled, of a formal series solution of F, $y_i = \alpha_i + v_i c^{h_i}$, $i = 2, \cdots, n$, $y_1 = \alpha_1 + \sum \varphi_k c^{\rho_k}$, where the summation is over the ordinals less than some ordinal ω, and the ρ_k are positive-valued monotonically increasing functions of the subscript. *W. Strodt.*

Theory of Groups

Hamill, C. M. On a finite group of order 6, 531, 840. Proc. London Math. Soc. (2) 52, 401–454 (1951).

This paper, submitted for publication in June 1948 and only now appearing in print, contains the large body of fundamental results on which four other papers, already published before its appearance, were based [E. M. Hartley, Proc. Cambridge Philos. Soc. 46, 91–105, 555–569 (1950); J. A. Todd, ibid. 46, 73–90 (1950); Proc. Roy. Soc. London. Ser. A. 200, 320–336 (1950); these Rev. 11, 578; 12, 241, 479]. The group G with which all these papers are concerned is a collineation group of S_5, of order $2^9\cdot 3^6\cdot 5\cdot 7$, which was first described by Mitchell. It is generated by 126 homologies, here called projections, and its subordinate collineation groups in the primes of the projections are of the type, generated by 45 projections, which was studied in detail by H. F. Baker [A locus with 25920 linear self-transformations, Cambridge Univ. Press, 1946; these Rev. 8, 400].

The main results of the paper are (i) a complete and concise description of the configuration formed by the 126 vertices; (ii) an analysis, remarkable for its simplicity and elegance, of the operations of the group, their classification into 31 types and 34 conjugate sets; and (iii) a similar analysis of the simple sub-group G' of G, of index 2, of which the operations are generated by products of an even number of the 126 projections.

As regards the configuration, the numbers of different types of line, plane, solid and prime that fall to be considered are 2, 4, 7 and 7 respectively; all these spaces are enumerated and an incidence table is given. As regards the analysis of operations of G, the author exhibits all of them as products of at most 6 of the projections; most of the 31 types are obtained by systematic examination of such operations as are products of projections from sets of vertices which lie on the various types of lines, planes, solids and primes of the configuration, the three types which then remain being dealt with separately. The numbers of operations of each type, their periods, the character of their powers, the numbers of projections required to generate them, the number of conjugate sets for each type (always 1 or 2) are exhibited in a comprehensive table, and this also gives corresponding information concerning the simple subgroup G'.

J. G. Semple (London).

Piccard, Sophie. Les bases du groupe symétrique dont l'une des substitutions est un cycle du sixième ordre. Comment. Math. Helv. 25, 91–130 (1951).

The author continues her investigation into the possible pairs of operations (S, T) which generate the symmetric group S_n; T is here taken to be a cycle of order six. In the first half of the paper the case where both S and T are cycles of order six is studied in detail. This case turns out to be crucial for the general problem. In fact, it appears that if a substitution group G which is transitive and primitive on n symbols contains two connected, imprimitive and independent cycles S, T of order six then G must be the symmetric group S_n. For $n\geqq 9$, the necessary and sufficient condition that S, $T=(123456)$ should generate S_n is that S and T be connected and primitive. The paper concludes with a listing of the possibilities for $n=6, 7, 8$.

G. de B. Robinson (Toronto, Ont.).

Cherubino, Salvatore. Gruppi abeliani di omografie piane. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 177–188 (1950).

The author discusses in detail the abelian subgroups of the one and two dimensional projective groups over the complex numbers. *S. A. Jennings* (Vancouver, B. C.).

Rado, R. A proof of the basis theorem for finitely generated Abelian groups. J. London Math. Soc. 26, 74–75; erratum, 160 (1951).

A short elementary proof of the basis theorem is given.

S. A. Jennings (Vancouver, B. C.).

Szele, T. On direct sums of cyclic groups. Publ. Math. Debrecen 2, 76–78 (1951).

Let G be an additive abelian group, and let $f(a)$ be the order of the element $a\epsilon G$, $1\leqq f(a)\leqq\infty$. Two finite sets of elements $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$ are equivalent if $\{a_1, \cdots, a_k\}=\{b_1, \cdots, b_k\}$. A set S of elements $\neq 0$ is an "extremal system" if S contains no finite subset of elements $a_1, \cdots, a_k$ equivalent to a system $b_1, b_2, \cdots, b_k$ in G such that

$$\min_{1\leqq i\leqq k} f(b_i) < \min_{1\leqq i\leqq k} f(a_i).$$

If $\{S\}=G$ then S is an extremal generating system. Using a lemma of Rado [see the preceding review] the author proves the following: If G has an extremal generating system S, then G is the (not necessarily finite) direct sum of cyclic groups. This result generalises the familiar basis theorem for finitely generated abelian groups. *S. A. Jennings.*

Szele, Tibor. Sur les groupes ayant un sous-groupe parfait. Bull. Sci. Math. (2) 74, 207–209 (1950).

A group is perfect if it coincides with its commutator subgroup. The author proves that a group having a perfect subgroup has also a (not necessarily proper) normal perfect subgroup. *S. A. Jennings* (Vancouver, B. C.).

Lekkerkerker, C. G. Lecture in the Actualiteiten series on the Minkowski-Hajós theorem. Math. Centrum Amsterdam. Rapport ZW 1951-008, 13 pp. (1951). (Dutch)

An exposition of the theorem of the title, following Rédei's simplification [Acta Univ. Szeged. Sect. Sci. Math. 13, 21–35 (1949); these Rev. 10, 683]. See also Rédei [Comment. Math. Helv. 23, 272–282 (1949); these Rev. 11, 318].

W. J. LeVeque (Manchester).

Kertész, A. On groups every subgroup of which is a direct summand. Publ. Math. Debrecen 2, 74–75 (1951).

A necessary and sufficient condition that a group G have the property that every subgroup H of G is a direct summand of G is that G be abelian and that each element of G have (finite) squarefree order. For this conclusion it is sufficient to require only that every cyclic subgroup of G shall be a direct summand. *R. M. Thrall.*

Brahana, H. R. Finite metabelian groups and the lines of a projective four-space. Amer. J. Math. 73, 539–555 (1951).

Of all metabelian groups generated by 5 elements, with the centre C coincident with the commutator subgroup, and with all elements (except the identity) of order p, there is one, G, such that any other is isomorphic to a maximal quotient group of G [Brahana, Amer. J. Math. 62, 365–379 (1940); these Rev. 1, 257]. Every element of G is of the form $cU_1^{x_1}\cdots U_5^{x_5}$, where $c\epsilon C$, U_i are the generators, and $x\epsilon GF(p)$.

This element is correlated with the point $(x_1, x_2, \cdots, x_4)$ in a finite projective 4-space X; the commutator of two elements then correlates with the line joining their points. The lines of X may be correlated with points in a 9-space S. The subgroups of C then correlate with the subspaces of S. A classification of the points, lines, planes, and 3-spaces of S is given. This confirms the completeness of the list given by J. A. Todd [Proc. London Math. Soc. (2) 30, 513–550 (1930)]. *H. A. Thurston* (Bristol).

Golovin, O. N. The metabelian products of groups. Mat. Sbornik N.S. 28(70), 431–444 (1951). (Russian)

Explanation of notation and preliminary and related results are to be found in two papers of the same author: [1] Doklady Akad. Nauk SSSR N.S. 58, 1257–1260 (1947); [2] Mat. Sbornik N.S. 27(69), 427–454 (1950); these Rev. 9, 493; 12, 672]. The first nilpotent (see [2]) or metabelian product of a class $\{A_\alpha\}$ of groups, defined by $\prod_\alpha^0 A_\alpha = F/(F, (A_\alpha))$ where F is the free product of the A_α, and the metabelian commutator

$$(A_\alpha)_Z = [(A_\alpha)\cdot(F, (A_\alpha))]/(F, (A_\alpha)),$$

an object which turns out to lie in the center of $\prod^0$, are studied in this paper. The orders of the elements of $\prod^0$ are determined; and by using a type of mapping on the set of groups A_α, a type similar to that given by Levi [J. Indian Math. Soc. (N.S.) 8, 78–91 (1944); these Rev. 7, 113], the form of $(A, B)_Z$ is found for some groups A and B. The mappings mentioned above and related products of groups are developed in detail and are used to discuss the general case of $(A_\alpha)_Z$. In particular, the structure of $(A_\alpha)_Z$ is determined if each $A_\alpha/(A_\alpha, A_\alpha)$ splits into a direct product of cyclic groups. A necessary and sufficient condition for a regular product to be a metabelian product (regular product as defined in [2]) is derived and is used to obtain the associativity of the metabelian product independently of the more general proof in [2] for the associativity of the nilpotent products. The derivative of a metabelian product $\prod_\alpha^0 A_\alpha$ is exhibited as the direct product of the derivatives of the factors A_α and of (A_α).

F. Haimo (St. Louis, Mo.).

Golovin, O. N. On the isomorphism of nilpotent decompositions of groups. Mat. Sbornik N.S. 28(70), 445–452 (1951). (Russian)

Preliminary material may be found in three earlier papers of the author [Doklady Akad. Nauk SSSR (N.S.) 58, 1257–1260 (1947); Mat. Sbornik N.S. 27(69), 427–454 (1950); these Rev. 9, 493; 12, 672; the paper reviewed above]. The author proves that a group which is the free product of a set of groups cannot be any nilpotent product of the same factors. In addition, he shows that if a finite group G is the kth nilpotent product of a finite number of primary cyclic groups, then this decomposition is essentially the only one of the kth class. It is proved that if a group coincides with its derivative or if it is centerless, then each of its nilpotent decompositions is a direct decomposition. The nilpotent products of a set of groups A_α all reduce to the direct product of these same factors if each $A_\alpha/(A_\alpha, A_\alpha)$ is periodic and if each has a characteristic (as defined, say, in the second reference) such that these characteristics are relatively prime in pairs; or, if all but two of these factor groups are periodic and complete, and of the remaining two, one is periodic and the other is complete. Related results and associated open questions are discussed.

F. Haimo (St. Louis, Mo.).

Chen, Kuo-Tsai. Integration in free groups. Ann. of Math. (2) 54, 147–162 (1951).

If F is a free group and E a vector space over the reals whose dimension equals the rank of F, then there exist fairly obvious homomorphic mappings σ of F into the group of translations of E such that the commutator subgroup $F' = [F, F]$ is the kernel of σ. If u is in F, and if f is a continuous real-valued function defined over E, then the product fu is defined by the rule: $(fu)(x) = f(xu^\sigma)$ for x in E. Finally an integral is defined as a real-valued function of the pairs (f, u) for u in F and f a real-valued continuous function over E which meets the following requirements:

(a) $$I(af+bg, u) = aI(f, u) + bI(g, u);$$
(b) $$I(f, uv) = I(f, u) + I(fu, v).$$

Clearly $I(f, u) = 0$ for every u in $F'' = [F', F']$. Thus integrals may be used for an investigation of F/F'' and more generally of G/G'' for any group G. This the author does. He determines the factors of the descending center chain of F/F'', gives a method for an explicit computation of the factors of the descending center chain of G/G'' for finitely generated groups G and applies this method on the groups of certain links. Of his explicit results we mention only the following ones. If F is a free group of finite rank n, then the dth factor of the descending central chain of F/F'' is a free abelian group of rank $(d-1)\binom{d+n-2}{d}$; and if F is a free group of infinite rank then the factors of the descending central chain of F/F'' are free abelian groups of infinite rank. Finally, a relation between Fox's differentiation and the author's integration is established. *R. Baer.*

Cockcroft, W. H. The word problem in a group extension. Quart. J. Math., Oxford Ser. (2) 2, 123–134 (1951).

Let Q and K be two groups which are given in terms of generators and relations. Let there be given a group E, containing K, and a homomorphism τ of E onto Q whose kernel is K. A choice of representatives $u(x)\epsilon E$ for the elements $x\epsilon Q$ enables one to specify a set of generators and relations defining E. It is shown here in detail that a solution of the word problem for K and Q gives a solution of the word problem for E.

Let X be a topological space with a finitely generated abelian fundamental group γ. Let X^* be the space obtained by attaching the boundaries of a finite number of oriented 2-cells to oriented circuits in X. The above method is combined with methods and results of J. H. C. Whitehead [Bull. Amer. Math. Soc. 55, 453–496 (1949); Ann. of Math. (2) 42, 409–428 (1941); (2) 47, 806–810 (1946); these Rev. 11, 48; 2, 323; 8, 167] to give an effective calculation of the second relative homotopy group $\rho = \pi_2(X^*, X)$, regarded here as an extension of $\rho/[\rho, \rho]$ by $[\rho, \rho]$, of the kernel of the natural homomorphism of ρ into γ, and of the γ-operators in ρ. *G. Hochschild* (New Haven, Conn.).

Brenner, J. L. The n^2 unitary group. Revista Ci., Lima 52, nos. 3–4, 9–14 (1950).

Let $n = v_1 + v_2 + \cdots + v_s$ be a partition of n, account being taken of the order of the parts, i.e., (2, 1) for this purpose being regarded as a different partition from (1, 2). The group of n^2 unitary matrices has a subgroup which consists of matrices of the form $A = \operatorname{diag}(A_1, A_2, \cdots, A_s)$ where A_r is a v_r-rowed unitary matrix. The author proves that the set of all such subgroups for all partitions of n forms a lattice, i.e., the meet and the union of any two such subgroups are themselves subgroups of the set. A group of matrices is algebraic if a necessary and sufficient condition that $A = [a_{st}]$

belongs to the group may be expressed by the vanishing of a set of polynomials $F(a_{11}, \overline{a_{11}}, a_{12}, \overline{a_{12}}, \cdots, a_{nn}, \overline{a_{nn}})$ belonging to a certain ideal. The above subgroups are clearly algebraic and for these particular groups the union of two algebraic group is itself an algebraic group. The author gives a proof which he ascribes to Zariski that for all groups of matrices the union of two algebraic groups is itself algebraic.
D. E. Littlewood (Bangor).

Tits, J. Les groupes projectifs: évolution et généralisations. Bull. Soc. Math. Belgique 3 (1949–1950), 1–10 (1951).

This is a resumé of the author's work on triply and n-tuply transitive groups [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 197–208, 224–233, 568–589, 756–773 (1949); these Rev. **11**, 9, 320], together with remarks of an historical nature concerning the important case of the projective groups.
L. M. Blumenthal (Los Angeles, Calif.).

Nakayama, Tadasi. Finite groups with faithful irreducible and directly indecomposable modular representations. Proc. Japan Acad. **23**, no. 3, 22–25 (1947).

Let K be a field of characteristic p, let $\mathfrak{G}$ be a finite group. Necessary and sufficient conditions are given that $\mathfrak{G}$ has a faithful representation which is (i) irreductible or (ii) indecomposable or (iii) an indecomposable component of the regular representation; or that $\mathfrak{G}$ has a faithful representation which is a direct sum of t components of each of these types (in turn). These conditions are formulated in several ways and are natural generalizations of those given for the nonmodular case by Shoda [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I, 2 (1930), 51–72 (1934); correction **2** (1931), 203–209 (1934)]. *R. M. Thrall* (Ann Arbor, Mich.).

Chung, J. H. Modular representations of the symmetric group. Canadian J. Math. **3**, 309–327 (1951).

According to a conjecture of Nakayama [Jap. J. Math. **17**, 165–184, 411–423 (1941); these Rev. **3**, 195, 196; **4**, 340] later proved by R. Brauer [Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 11–19 (1947); these Rev. **10**, 678] and G. de B. Robinson [ibid. **41**, 20–25 (1947); these Rev. **10**, 678] the block structure of the modular group algebra of the symmetric group is completely determined by the theory of p-hooks and p-cores. The modular theory for blocks of defect 0 is the same as the non-modular theory. For blocks of defect 1 Nakayama determined the decomposition number (i.e., the invariants which describe the way in which an ordinary irreducible representation splits when considered modulo p) subject to the restriction $n<2p$. The author now removes this restriction on n. Another major result in the present paper is the complete determination of the decomposition numbers for ζ_{2p} ($n=2p$ is the smallest value of n for which ζ_n has blocks of defect 2). The major part of the paper is devoted to the derivation of certain identities between the characters in a p-block of ζ_n. These identities are not independent and certain linear relations between them are obtained. These identities give as a special case certain identities between the decomposition numbers of ζ_n. *R. M. Thrall* (Ann Arbor, Mich.).

Mackey, George W. On induced representations of groups. Amer. J. Math. **73**, 576–592 (1951).

The author states that it is the purpose of the first part of the article to unify a part of the work of Frobenius and his successors by showing that a number of their results are easy consequences of a single theorem (Theorem 2) about Kronecker products of induced representations. Indeed, Theorem 2 contains as special cases the classical Frobenius reciprocity theorem, Shoda's results [Proc. Phys. Math. Soc. Japan (3) **15**, 249–257 (1933)] on the irreducibility and equivalence of monomial representations and Artin's theorem [Abh. Math. Sem. Univ. Hamburg **3**, 89–108 (1923)] on the expressibility of a given character as a rational linear combination of certain induced characters. [Reviewer's remark. It would be interesting to know whether the methods of the present paper could be used to obtain also Brauer's theorem [Ann. of Math. (2) **48**, 502–514 (1947); these Rev. **8**, 503] on the expression of a given character as an integral linear combination of certain special induced characters and perhaps throw new light on this kind of problem.

In part II induced representations of separable locally compact groups are studied as defined by the author [Proc. Nat. Acad. Sci. U. S. A. **35**, 537–545 (1949); these Rev. **11**, 158]. The results are extensions of part I to the infinite-dimensional case. Thus the author obtains expressions for the intertwining number of induced representations, which contain as special case a generalization of the Frobenius reciprocity theorem to the infinite dimensional case. This is done in the present paper under such assumptions as avoid rather complicated topological and measure theoretic difficulties and bring out more the algebraic aspect of the problems. The more general case is to appear elsewhere.

In part III the author presents certain applications of part II. Among others he obtains a criterion which allows one to establish pathological properties of the regular representation of certain infinite discrete groups rather directly. Some of the results have been announced earlier [C. R. Acad. Sci. Paris **230**, 808–809, 908–909 (1950); these Rev. **11**, 580, 713]. *F. I. Mautner* (Baltimore, Md.).

Harish-Chandra. Representations of semisimple Lie groups on a Banach space. Proc. Nat. Acad. Sci. U. S. A. **37**, 170–173 (1951).

Let G be a connected Lie group and H a Banach space. The author studies continuous representations π of G by bounded operators $\pi(x)$ $(x\epsilon G)$ on H. Let g_0 be the Lie algebra of G, g the complexification of g_0 and B the universal (associative) enveloping algebra of g. According to Gårding [same Proc. **33**, 331–332 (1947); these Rev. **9**, 133] there exists a dense linear subspace V of H such that π induces (by differentiation) a representation π_V of g_0 by linear transformations of V, which can be extended uniquely to a representation of B which is again denoted by π_V. Unfortunately, the closure in H of a linear subspace of V which is invariant under the operators $\pi_V(b)$ $(b\epsilon B)$ need not be invariant under the operators $\pi(x)$. To overcome this difficulty the author introduces a certain linear subspace W of H whose elements he calls well behaved. Derivatives along one-parameter subgroups of G exist for the elements ψ of W. Also for any continuous linear functional f on H the function $f(\pi(x)\psi)$ of x is analytic on G. It follows that the closure in H of a B-invariant linear subspace of W is G-invariant.

Now let G^* be the adjoint group of G, K^* a maximal compact subgroup of G^* and K the complete inverse image of K^* in G. Let D be an equivalence class of finite-dimensional irreducible representations of K and H_D the subspace of those elements of H which transform according to D under K. Theorem 2 asserts that the finite linear combinations of the elements of the various W_D are dense in H and that H_D

is the closure of W_D. Thus in this sense there exist sufficiently many well behaved elements. A representation is called quasi-simple if both the centers of G and B are mapped onto scalars. Theorem 3 asserts that every finite-dimensional irreducible representation of K occurs with finite multiplicity in any quasi-simple representation of G. From this it follows (theorem 4) that if π is quasi-simple then there exist two closed invariant subspaces $H_1 \subset H_2$ of H such that H_1 is proper maximal invariant in H_2. This has the very important consequence that all continuous unitary representations of G (in a Hilbert space) generate rings of operators of type I in the sense of Murray and von Neumann. It is known that this implies (among other things) that the Plancherel formula for $L_2(G)$ is in terms of the ordinary trace of operators of the Hilbert-Schmidt class. Theorem 6 asserts that there is only a finite number of inequivalent irreducible unitary representations of G which contain a given irreducible representation of K and coincide on the center Z of the enveloping algebra B.

Now let π be irreducible unitary in a Hilbert space H and E_D the orthogonal projection of H onto H_D. Since by theorem 4 $\dim H_D < \infty$, the function $\varphi_D{}^\pi(x) = \operatorname{Trace} E_D\pi(x)E_D$ is well defined for all $x\varepsilon G$. These functions constitute one of several possible generalizations of the spherical functions of Gelfand and Naimark (who treated the case $\pi(x)=1$). According to theorem 7 the functions $\varphi_D{}^\pi(x)$ determine π in a certain sense. *F. I. Mautner* (Baltimore, Md.).

Harish-Chandra. Representations of semisimple Lie groups. II. Proc. Nat. Acad. Sci. U. S. A. **37**, 362–365 (1951).

This is a continuation of the paper reviewed above. Using the same notation as in the preceding review, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. According to a result given in a previous paper of the author [Trans. Amer. Math. Soc. 70, 28–96 (1951)] each homomorphism χ of the center Z of B into the complex numbers is determined in a certain manner by a complex linear functional Λ on $\mathfrak{h}$ and denoted by χ_Λ. If π is a quasi-simple representation of G, the corresponding homomorphism of Z is now called the infinitesimal character χ of π. Theorem 1 discusses the problem of characterizing π by means of χ and one highest weight λ of K occurring in π. Also, for a given χ and λ there exists a corresponding quasi-simple irreducible representation on a Hilbert space. The note closes with a more detailed study of the functions $\varphi_D{}^\pi(x)$ introduced above. Theorem 2 is an improvement of theorem 7 of the preceding review and theorem 3 gives an explicit formula for $\varphi_D{}^\pi(x)$ as an integral over K^* for the case $\dim D = 1$. *F. I. Mautner* (Baltimore, Md.).

Harish-Chandra. Representations of semisimple Lie groups. III. Characters. Proc. Nat. Acad. Sci. U. S. A. **37**, 366–369 (1951).

The author continues the study of representations π of semi-simple Lie groups G. Using the same notation and terminology as in the two preceding reviews, theorem 1 gives an upper bound for the dimension of H_D. Theorem 2 treats the problem of associating with an irreducible quasi-simple representation π of G on a Hilbert space a character T_π defined globally on G as follows. If $f(x)$ is an infinitely differentiable function on G with compact support, then the operator $\int_G f(x)\pi(x)dx$ has a trace (as a finite or infinite dimensional matrix). This trace is denoted by T_f and turns out to be a distribution (in the sense of L. Schwartz), invariant under inner automorphisms of G. Theorem 3 asserts that T_π determines π in a certain sense, which coincides with ordinary equivalence in the case of irreducible unitary representations. The paper closes with the description of an explicit formula for T_π which generalizes both H. Weyl's formula for the characters of compact semi-simple Lie groups and the formula given for T_π by Gelfand and Naimark for the complex unimodular group. Thus the author has succeeded in extending a considerable part of the classical representation theory of semi-simple Lie groups of E. Cartan and H. Weyl to the infinite-dimensional case. His results contain also as a special case part of the work of Gelfand and Naimark on the unitary representations of the complex unimodular group. *F. I. Mautner.*

Harish-Chandra. Representations of semisimple Lie groups. IV. Proc. Nat. Acad. Sci. U. S. A. **37**, 691–694 (1951).

This note is a continuation of three earlier ones [see the three preceding reviews]. Theorem 1 deals with the relationship between certain maximal left ideals in the associative enveloping algebra of the given semi-simple Lie algebra and certain irreducible representations of the Lie group G on a Hilbert space. Theorem 2 extends results announced by the author in the earlier notes about the relationship between representations of G and of K, where K is the complete inverse image of K^* under the natural homomorphism $G \to G^*$ of G onto its adjoint group G^* and K^* a maximal compact subgroup of G^*. In particular the spherical functions are generalized to arbitrary irreducible representations of K and an almost explicit formula is given for them. Theorem 3 deals with the possibility and uniqueness of obtaining, from certain irreducible representations of G, unitary irreducible representations by introducing a new inner product. In a footnote the author mentions some minor errors and misprints in the earlier notes. *F. I. Mautner.*

Bonneval, Georges. Sur la topologie du groupe des rotations dans l'espace. Revue Sci. **89**, 83–89 (1951).

A proof of the theorem that the group of rotations of Euclidean three-space is doubly-connected. *S. Chern.*

***Cartan, Henri. Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 15–27. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

***Cartan, Henri. La transgression dans un groupe de Lie et dans un espace fibré principal.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 57–71. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

A principal fiber space for a connected Lie group G is a fiber space $\mathcal{E}$ whose fibers are isomorphic to the whole group G; thus, G operates (on the right) on $\mathcal{E}$. The two papers being reviewed are concerned with the mutual relations which exist between the cohomologies of a principal fiber space $\mathcal{E}$, of its base space $\mathcal{B}$ and of the group G. The author first extracts from the topological set-up of the problem a purely algebraic structure; the study of this structure is then made to yield topological results.

Let E be the algebra of differential forms of class C^∞ on $\mathcal{E}$. Then E has a structure of graded algebra (corresponding to the notion of degree of a differential form), in which we have the commutation rule $ab = (-1)^{pq}ba$, if a and b are homogeneous of degrees p and q. The algebra E has also a structure of differential algebra, defined by the operator d of derivation of differential forms; d is an antiderivation (i.e. $d(ab) = (da)b + (-1)^p a(db)$, if a is homogeneous of degree p).

Moreover, the fibered structure of $\mathcal{E}$ gives rise to certain relations between the Lie algebra $\mathfrak{a}(G)$ of G and the algebra E. To every $x \in \mathfrak{a}(G)$ there corresponds an infinitesimal transformation $T(x)$ on $\mathcal{E}$ which is defined as follows: x may be regarded as the infinitesimal transformation of a one-parameter subgroup $G(x)$ of right translations in G; since G operates on $\mathcal{E}$ on the right, $G(x)$ gives rise to a one parameter group of transformations of $\mathcal{E}$ into itself, and $T(x)$ is the infinitesimal transformation of this group. Now, every infinitesimal transformation on $\mathcal{E}$ defines two operators on E, a derivation θ (i.e. $\theta(ab)=(\theta\cdot a)b+a(\theta\cdot b)$) which is homogeneous of degree 0 (i.e. it preserves degrees) and an antiderivation i, which is homogeneous of degree -1. If $x \in \mathfrak{a}(G)$, let $\theta(x)$ and $i(x)$ be the operators on E which correspond to $T(x)$. These operators and the differential operator d are bound together by the formulas

(I) $\theta([x, y])=\theta(x)\theta(y)-\theta(y)\theta(x)$;

(II) $\theta(x)i(y)=i(y)\theta(x)+i([x, y])$; (III) $\theta(x)=i(x)d+di(x)$ (for x and y in $\mathfrak{a}(G)$).

If P is a point of $\mathcal{E}$, then some neighbourhood of the fiber of P may be represented as the product of G and a neighbourhood of the projection Q of P on the base space $\mathfrak{B}$. It follows that we have, for every point P' of this neighbourhood, a representation of the tangent space to $\mathcal{E}$ at P' as the direct sum of the tangent space to G (i.e. of $\mathfrak{a}(G)$) and of the tangent space to $\mathfrak{B}$ at the projection of P'. This decomposition is of course not intrinsically defined. Nevertheless, it has been proved by C. Ehresman and A. Weil that it is possible to define such a decomposition of the tangent space to $\mathcal{E}$ which is defined and of class C^∞ all over $\mathcal{E}$ and which is invariant relatively to the operations of G. Such a decomposition is called a "connexion" in $\mathcal{E}$. It defines at every point P of $\mathcal{E}$ a linear mapping of the tangent space to $\mathcal{E}$ at P onto the Lie algebra $\mathfrak{a}(G)$ of G, or, by duality, a linear isomorphism of the dual space $\mathfrak{a}^*(G)$ of $\mathfrak{a}(G)$ into the space E^1 of differential forms of degree 1 on $\mathcal{E}$. Let $A(G)$ be the exterior algebra on $\mathfrak{a}^*(G)$, i.e. the algebra of left invariant differential forms on G. Then f extends to a homogeneous homomorphism f of degree 0 of $A(G)$ into E which satisfies the following conditions: (IV) $i(x)\cdot f(a)=f(i(x)\cdot a)$; (V) $\theta(x)\cdot f(a)=f(\theta(x)\cdot a)$. In these formulas, $i(x)$ and $\theta(x)$ denote on the left side the operators which are associated to an $x \in \mathfrak{a}(G)$ in E, while on the right side they denote the operators associated to x in $A(G)$, which are defined in the same manner as those on E by considering G as a fiber space over itself.

The cohomology of the space $\mathcal{E}$ (with real coefficients) is the same as that of E. If G is compact, then it is also the cohomology of the subalgebra I_E of E composed of the invariant elements a (i.e. the elements a such that $\theta(x)\cdot a=0$ for all x in $\mathfrak{a}(G)$). Furthermore, the cohomology of $\mathfrak{B}$ is the cohomology of the algebra B of basic elements of E (i.e. of elements a such that $\theta(x)\cdot a=i(x)\cdot a=0$ for all x in $\mathfrak{a}(G)$).

At this stage, topology may be thrown out, leaving only the algebraic facts in evidence. We assume that there is given a graded differential algebra E on a field K of characteristic 0, in which the usual commutation formula holds true, and a Lie algebra $\mathfrak{a}$ over K which "operates" on E, in the sense that, to every x in $\mathfrak{a}$, there are associated in a linear manner linear operators $\theta(x)$ and $i(x)$ on E, a derivation of degree 0 and an antiderivation of degree -1, respectively, for which the formulas (I), (II), (III) hold true. Furthermore, it is assumed that there exists a "connexion" in E, i.e. an isomorphism f of the exterior algebra A over the dual of $\mathfrak{a}$ for which the formulas (IV), (V) hold true. It is to be observed that only the existence of at least such a connexion is postulated; no specific connexion is assumed to be given.

The algebra A is itself a differential algebra; let d be its differential operator. If f is a connexion, we do not have in general $f\circ d=d\circ f$. The mapping $\varphi: x^*\rightarrow df(x^*)-f(dx^*)$, x^* in the dual of $\mathfrak{a}$, is called the "curvature" of the connexion. It transforms the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ into a subspace of the center of E, and may therefore be extended to a homomorphism φ of the symmetric algebra S on $\mathfrak{a}^*$ into E. We have now two homomorphisms: f of A into E and φ of S into the center of E. It is natural to combine these two homomorphisms into one by introducing the tensor product $W=A\otimes S$ of A and S: f and φ, taken together, define a homomorphism F of W into E. The algebra W is called the "Weil algebra" of the Lie algebra $\mathfrak{a}$. In order for F to conserve degrees, one defines a gradation in W in which the elements of A have their natural degrees, while to every element of S there is attributed twice its natural degree. It is possible to make the Lie algebra $\mathfrak{a}$ "operate" on W (in the sense indicated above) and to define in W a differential operator δ in such a way that the homomorphism F of the algebra W into E is also a homomorphism of the full structure defined by the operators of $\mathfrak{a}$ and by the differential operator. The manner in which $\mathfrak{a}$ operates on W and the differential δ are defined in terms of $\mathfrak{a}$ alone, without reference to E. The basic elements of W turn out to be exactly those elements of S which are invariant; they form a subalgebra I_S of S which is mapped by F into the algebra B of basic elements of E. The operator δ is zero on I_S, which shows that I_S is its own cohomology algebra. Thus, there corresponds to F a homomorphism F^* of I_S into the cohomology $H(B)$ of B (i.e. into the cohomology of the base space, in the case where E comes from a principal fiber space). The first main theorem of the paper states that, although the homomorphism F depends on the choice of the connexion f, the mapping F^* does not. The image $F^*(I_S)$ is therefore an intrinsically defined subalgebra of $H(B)$; this subalgebra is called the "characteristic subalgebra", and its elements the "characteristic cohomology classes"; their degrees are even.

The first main theorem is proved by introducing the tensor product $\bar{E}=E\otimes W$ of E and W (this is not quite the ordinary tensor product; the multiplication rule is modified in such a way as to preserve the commutation formula $ba=(-1)^{pq}ab$). The Lie algebra $\mathfrak{a}$ still operates on $\bar{E}$, and $\bar{E}$ is a differential algebra. Let $\bar{B}$ be the algebra of basic elements of $\bar{E}$. Then $\bar{B}$ contains B and I_S, and we therefore have natural mappings $H(B)\rightarrow H(\bar{B})$, $I_S\rightarrow H(\bar{B})$. The author proves that the first one of these mappings is an isomorphism onto; making use of its inverse, we obtain a natural mapping $I_S\rightarrow H(B)$, whose definition does not depend on the choice of a connexion, and which is proved to be identical to F^*, when F is the homomorphism defined by an arbitrary connexion.

The algebra $\bar{E}$ is $E\otimes A\otimes S$; the author now introduces the subalgebra $E\otimes S$ of $\bar{E}$. Let C be the algebra of invariant elements of $E\otimes S$. A certain differential Δ is defined on C, and it is proved (this is the second main theorem) that $H(B)=H(\bar{B})$ may be identified in a natural manner to the cohomology $H(C)$ of C with respect to the operator Δ. In order to accomplish the next step, it is necessary to assume that the Lie algebra $\mathfrak{a}$ is reductive (i.e. that its adjoint representation is semi-simple; this is always the case when $\mathfrak{a}$ is the Lie algebra of a compact group), and that E is finite-

dimensional (or may be replaced for all practical purposes by a finite-dimensional algebra, as happens in the case of a principal fiber space). The author then applies what he calls the Hirsch-Kozul theory [cf. G. Hirsch, C. R. Acad. Sci. Paris **227**, 1328–1330 (1948); these Rev. **10**, 558]. This gives the following result: the cohomology space (not algebra) of B may be identified with that of the vector space $H(E)\otimes I_S$, on which a suitable coboundary operator D has been defined. This operator D is 0 on I_S and maps $H(E)$ into the subset I_S^+ of I_S spanned by the homogeneous elements of degrees >0; it is not defined intrinsically.

This last reduction gives in particular results on those fiber spaces $\mathcal{E}$ which are "classifying" up to a certain dimension N, which means that $H_i(E)=\{0\}$ for $1\leqq i\leqq N$, $H_0(E)$ being the field of real numbers. In that case, $H_m(B)=\{0\}$ for every odd $m>0$ which is $\leqq N$, while, if $0<2p\leqq N$, $H_{2p}(B)$ is isomorphic to the space I_S^p of homogeneous elements of degree p of I_S.

It is possible to go further in the case where the fiber space $\mathcal{E}$ is itself a group, of which G is a subgroup; the base space $\mathcal{B}$ is then a homogeneous space for the big group $\mathcal{E}$. In order to study this situation, one has to introduce the operation of transgression in the Weil algebra W of a Lie algebra $\mathfrak{a}$. The definition of this operation is based on a theorem to the effect that the cohomology of the differential operator δ of W is trivial. Let then u be an element of I_S^p ($p\geqq 1$); since δ is 0 on I_S, u may be written in the form δw, w in W. Since u must be considered to be of degree $2p$ in W, w is of degree $2p-1$; it may be assumed to be invariant. Since $W=A\otimes S$, w may be projected on an element w_A of A, and w_A belongs to the algebra I_A of invariant elements of A. The mapping $u\to w_A$ of I_S^p into I_A will be denoted by ρ. The elements of $\rho(I_S^p)$ are called "transgressive", and, the notation being as above, w is called a transgression cochain for w_A. Assume now that the algebra $\mathfrak{a}$ is reductive. In that case, the structure of I_A has been determined by Hopf and Koszul [cf. Koszul, Bull. Soc. Math. France **78**, 65–127 (1950); these Rev. **12**, 120]. It is the exterior algebra over a certain uniquely defined subspace P_A, which is called the space of "primitive" elements. The following theorems have been conjectured by A. Weil and proved partly by the author and partly by the reviewer: $\rho(I_S^p)$ is the space of primitive elements of degree $2p-1$ of I_A, and the kernel of ρ in I_S^p consists of those elements which may be written as sums of products of homogeneous elements of degrees >0 in I_S. It is therefore possible to define (not intrinsically) a mapping τ of P_A into I_S such that $\tau\circ\rho$ is the identity; such a mapping is called a "transgression".

This being said, the algebraic counterpart of the case where $\mathcal{E}$ is a group is the case in which E is the exterior algebra over the dual of a Lie algebra $\mathfrak{A}$. We then set $E=A(\mathfrak{A})$, while the algebra previously denoted by A is now denoted by $A(\mathfrak{a})$; and similarly for W, S, I_S, I_A. The cohomology algebra $H(B)$ is now denoted by $H(\mathfrak{A}/\mathfrak{a})$. It is assumed that $\mathfrak{A}$ is reductive, and that $\mathfrak{a}$ is not only reductive, but also reductive in $\mathfrak{A}$ (i.e. the adjoint representation of $\mathfrak{A}$ induces a semi-simple representation of $\mathfrak{a}$). The algebra $H(E)$ is now $I_A(\mathfrak{A})$, and $H(E)\otimes I_S$ is $I_A(\mathfrak{A})\otimes I_S(\mathfrak{a})$. The novelty of the present case consists in this that, while the multiplicative structure was lost at the last step of the analysis of the general case, it can here be preserved. The differential D in $I_A(\mathfrak{A})\otimes I_S(\mathfrak{a})$ whose cohomology gives $H(\mathfrak{A}/\mathfrak{a})$ may now be defined as follows. Select a transgression τ from $P_A(\mathfrak{A})$ to $I_S(\mathfrak{A})$, and compose it with the natural homomorphism of $I_S(\mathfrak{A})$ into $I_S(\mathfrak{a})$ whose existence results from the fact that $\mathfrak{a}$ is a subalgebra of $\mathfrak{A}$; in this way, we obtain a mapping of $P_A(\mathfrak{A})$ into $I_S(\mathfrak{a})$, which may be extended to an antiderivation D of the algebra $I_A(\mathfrak{A})\otimes I_S(\mathfrak{a})$; the cohomology algebra of D is then $H(\mathfrak{A}/\mathfrak{a})$. This shows that $H(\mathfrak{A}/\mathfrak{a})$ is entirely determined by the homomorphism $I_S(\mathfrak{A})\to I_S(\mathfrak{a})$.

There exists a natural homomorphism of $H(\mathfrak{A}/\mathfrak{a})$ into $H(\mathfrak{A})$; the image of $H(\mathfrak{A}/\mathfrak{a})$ by this homomorphism is generated by a certain subspace Q of the space $P_A(\mathfrak{A})$ (theorem of Samelson). The dimension of Q is at most equal to the difference $R-r$ between the ranks R of $\mathfrak{A}$ and r of $\mathfrak{a}$. The result quoted above relatively to $H(\mathfrak{A}/\mathfrak{a})$ gives very precise information in the case where the dimension of Q reaches its highest value $R-r$ (this obviously happens if $R=r$; it also happens when $\mathfrak{a}$ is the algebra of elements of $\mathfrak{A}$ which are left invariant by some automorphism of order 2; the latter case corresponds to the case of homogeneous spaces which are compact symmetric Riemann spaces). Assuming that the dimension of Q is $R-r$, $H(\mathfrak{A}/\mathfrak{a})$ may be represented as the tensor product of its characteristic subalgebra and of an other subalgebra which is mapped isomorphically into $I_{\mathfrak{A}}$ and which is therefore isomorphic to the exterior algebra over Q. Moreover, the homology theory of S-modules (due to Kozul; cf. the following review) then yields an explicit expression for the Poincaré polynomial of $H(\mathfrak{A}/\mathfrak{a})$.

The theory of transgression is also used by the author in studying the problem inverse to the one considered above, namely to reconstruct $H(E)$ when $H(B)$ is given. Let w_A be a primitive element of I_A and let w be a transgression cochain for w_A: thus, w is an invariant element of W whose projection on A is w_A and whose coboundary is in I_S. Let F be the homomorphism $W\to E$ which corresponds to a transgression; then $F(w)$ is a cochain in E which "induces" the cocycle w_A on every fiber and whose coboundary is a cocycle of the base space: this gives rise to a transgression from P_A into $H(B)$. Making use of this transgression, it is proved that $H(E)$ may be represented as the cohomology of a certain differential on the algebra $I_A\otimes B$; the construction of this differential depends on the mapping $I_S\to B$ defined by a connexion in E.

The two papers being reviewed (and the paper reviewed below) are written very concisely and very clearly; they contain some indications on the methods of proof of the theorems which are stated. Their content is not exhausted by the above review; in particular, we have not mentioned the analysis of what happens when the fiber space $\mathcal{E}$ over the group G is considered as a fiber space over a subgroup of G. The author also explains how the Weil algebra could be considered as the algebra of cochains of an (nonexistent) fiber space which would be classifying in all dimensions.

C. Chevalley (New York, N. Y.).

✓ ★**Koszul, J. L. Sur un type d'algèbres différentielles en rapport avec la transgression.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 73–81. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

The first part of this paper contains an exposition of what H. Cartan has called the Hirsch-Koszul theory [cf. the preceding review]. Let E be the algebra of differential forms on a principal fiber space $\mathcal{E}$; then E contains as a subalgebra the algebra B of differential forms on the base space $\mathcal{B}$ of $\mathcal{E}$. Let R_q be the ideal generated in E by the homogeneous elements of degree q of B; then the sequence of ideals R_q de-

fines a filtration of E. The algebraic situation considered by the author is the more general one in which the ideals R_q are given, rather than the algebra B. Let E be a graded differential algebra with a unit element, in which the usual commutation formula $ba=(-1)^{pq}ab$ holds true (a, b homogeneous of degrees p, q); let E be filtered by a descending sequence (R_q) of ideals [cf. H. Cartan, C. R. Acad. Sci. Paris 226, 148–150 (1948); these Rev. 9, 368]. It is then possible to construct a Leray-Koszul sequence for E, whose last term is the cohomology algebra of E. The first term of this sequence is an algebra L which is the direct sum of the spaces $L_q=H(R_q/R_{q+1})$. However, it is possible under certain conditions to introduce directly in L a coboundary operator (distinct from the Leray operator) whose cohomology space is that of E; the definition of this operator is the object of the Hirsch-Koszul theory. Let B_q be the space composed of all elements x of degree q of R_q such that dx is in R_{q+1}; the set $B=\sum B_q$ may be identified with a subalgebra of L. Let F be the subset of $L_0=H(E/R_1)$ composed of those cohomology classes in L_0 which contain a cochain which is a cocycle not only modulo R_1 but modulo R_2. Assume that the following condition is satisfied: (1) F is finite-dimensional, and any base of F is a set of free generators for the structure of module of L over its subalgebra B. It is then possible to extend to L the coboundary operator of B in such a way that the cohomology space of the extended operator be isomorphic to that of E.

An element of F is called "transgressive" if it contains not only a cocycle modulo R_2 but a cochain whose coboundary is in B [compare the preceding review]. Assume furthermore that the following condition is satisfied: (2) F is the exterior algebra over a homogeneous vector space P whose elements are transgressive and which is spanned by homogeneous elements of odd degrees. Then the extended coboundary operator in L may be taken to be an antiderivation, and its cohomology algebra is the cohomology algebra of E; this is a generalization of a theorem proved by the reviewer. The conditions (1), (2) are in particular satisfied when E is the algebra of differential forms on a principal fiber space.

Assume that conditions (1), (2) are satisfied. To every x in P there is associated a y in E such that $dy \in B$; the mapping $x \to dy$ may be extended to a homomorphism of the symmetric algebra S over P into the center of B, which confers on B a structure of algebra over S. This leads the author to a general study of homological properties of S-modules M, S being the symmetric algebra over any finite-dimensional vector space V. Let A be the exterior algebra over V; the author defines on $A \otimes M$ a coboundary operator D by the formula

$$D((x_1 \wedge \cdots \wedge x_p) \otimes b) = \sum (-1)^{i+1} (x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p) \otimes (x_i b)$$

(the x_i's being in V and b in M). The cohomology space of D is called the "cohomology space of the S-module" M and denoted by $H(M)$. The natural gradation of A defines a gradation $H(M)=\sum_p H_p(M)$ on $H(M)$. If M is not only a module but an algebra over S, then D is an antiderivation and $H(M)$ a graded algebra. From now on, it will be further assumed that S and M have gradations, called T-gradations, such that, when s is of degree p in S and b of degree q in M, then $s \cdot b$ is of degree $p+q$. Denote by S^+ the ideal generated in S by the homogeneous elements of degrees >0, and represent M as the direct sum of S^+M and a homogeneous vector space U_0; any base of U_0 will be a set of generators of the S-module M. There is a natural homomorphism φ_1 of the tensor product $S_1=S \otimes U_0$ onto M; the kernel M_1 of φ_1 may be considered to be the module of linear relations which exist between the elements of a base of U_0. The operation by which one passes from M to M_1 may be repeated indefinitely; in this way, the author constructs infinite sequences $(M_0, \cdots, M_p, \cdots)$, $(S_0, \cdots, S_p, \cdots)$ of S-modules and $(U_0, \cdots, U_p, \cdots)$ of vector spaces. This procedure is a generalisation of Hilbert's construction of the "abgeleitete Gleichungssysteme" of an ideal J in S, to which it reduces when $M=J$. The author proves the remarkable fact that for every p, U_p is isomorphic to $H_p(M)$. It follows in particular that $U_p=\{0\}$ when p is larger than the dimension of V.

The author defines a T-gradation in A by the condition that the elements of V, considered as generators of A, should have a degree one less than their degree in S. This gives a T-gradation in $A \otimes M$, and therefore also in $H(M)$; $H(M)$ therefore now has a double gradation. Let $a_{p,n}$ be the dimension of the space of elements of $H(M)$ of A-degree p and T-degree n; introduce the formal series $U(t, \theta)=\sum_{p,n} a_{p,n} \theta^p t^n$. Denote by μ_n, σ_n the dimensions of the spaces of T-degree n in M, S, and set $M(t)=\sum \mu_n t^n$, $S(t)=\sum \sigma_n t^n$. Then $M(t)$, $S(t)$ and $U(\theta, t)$ are related to each other by the identity $M(t)=S(t)\,U(t, -t)$.

In order to describe the applications of this theory, let us come back to the notation of the beginning of this review. Assuming that conditions (1), (2) are satisfied, the author considers the case in which every element of B is a cocycle. This happens in particular if E is the Weil algebra of a Lie algebra [cf. preceding review] and also in the case where E is the exterior algebra over the dual of a reductive Lie algebra $\mathfrak{A}$, B being the algebra of relative cochains with respect to a subalgebra $\mathfrak{a}$ of $\mathfrak{A}$ which is the set of elements left invariant by an automorphism of order 2 of $\mathfrak{A}$. Whenever this situation arises, $H(E)=H(L)$ turns out to be the cohomology algebra of the S-algebra B. In particular, when E is the Weil algebra of a reductive Lie algebra $\mathfrak{a}$, the cohomology of B must be trivial, and it follows from the theory of S-modules that this implies that B is isomorphic to S. But B is the algebra of invariant elements of the symmetric algebra over the dual of $\mathfrak{a}$; this proves the theorem (also obtained by the reviewer by entirely different methods) that the algebra of invariants of the adjoint representation of a reductive Lie algebra may be generated by algebraically independent invariants. The theory is also applied to the determination of the Poincaré polynomial of $H(\mathfrak{A}/\mathfrak{a})$ when $\mathfrak{A}$ is reductive and $\mathfrak{a}$ is the algebra of elements left invariant by an automorphism of order 2 of $\mathfrak{A}$. Taking H. Cartan's results into account [cf. preceding review], this part of Koszul results may be extended to a more general case; it leads in particular to a proof and to a generalisation of a formula conjectured by Hirsch which gives the Poincaré polynomial of $H(\mathfrak{A}/\mathfrak{a})$ when $\mathfrak{A}$ is reductive, $\mathfrak{a}$ is reductive in $\mathfrak{A}$ and has the same rank as $\mathfrak{A}$. *C. Chevalley.*

Vilenkin, N. Ya. On the classification of zero-dimensional locally compact periodic Abelian groups without elements of finite order. Mat. Sbornik N.S. 28(70), 503–536 (1951). (Russian)

This is a detailed paper on a substantial part of the author's extension of the theory of the discrete countable torsion groups. Some of the results were announced in an earlier note [Doklady Akad. Nauk SSSR (N.S.) 61, 969–971 (1948); these Rev. 10, 282] and part are contained in §10

of chapter II of a general expository paper [Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 19–74 (1950); these Rev. 12, 78]. The author uses the principal results and follows the notation of his first paper on the direct factoring of topological groups [Mat. Sbornik N.S. 19(61), 85–154 (1946); these Rev. 8, 132]. The groups dealt with here are zero-dimensional, locally compact abelian groups G (i.e. of type NL in the author's notation), primary (i.e. for some fixed prime p and every $g\epsilon G$, $p^n g$ converges to the identity with increasing integer n), with no elements of finite order (notation: ${}_{\infty}[G]=0$), and satisfying a condition known as completely proper stratification:

$$p^n G\cap \overline{p^{\alpha+n}G}=p^n\overline{[p^\alpha G]},$$

for every finite integer n and every transfinite α for which the corresponding subgroup of elements of infinite height is (inductively) defined in G. *L. Zippin.*

NUMBER THEORY

Katz, Alexander. Third list of factorizations of Fibonacci numbers. Riveon Lematematika 5, 13 (1951). (Hebrew. English summary)

Factors of Fibonacci's U_n for $n=141$, 147, 165, 189, and of $V_n=U_{2n}/U_n$ for $n=147$, 153, 180, 189, are given as 108289, 3529, 86461, 38933, 65269, 13159, 8641, 85429, respectively. Also V_{138} is completely factored and has the primitive prime factors 16561, 162563, 10437 66587. The factorizations of U_{147} and U_{165}, given by the author as possibly incomplete, are really complete. *D. H. Lehmer.*

Jarden, Dov. On sums of reciprocals of terms in arithmetical progression. Riveon Lematematika 5, 14–15 (1951). (Hebrew. English summary)

The theorem proved is the following. Let p^α be a power of a prime, let $p^\alpha=1+d$, and let (1) $b_k=p^\alpha+kd$ ($k=0, 1, \cdots, n$). Let b_μ be the term of this arithmetical progression which is equal to the highest power of p^α occurring in (1). Let $a_0, a_1, \cdots, a_n$ be an arbitrary sequence of integers subject only to the condition that a_μ is not divisible by p. Then the sum $\sum_{k=0}^n a_k/b_k$ is not an integer. This result is related to one by P. Erdös [Mat. Fiz. Lapok 39, 17–24 (1932)].

D. H. Lehmer (Berkeley, Calif.).

Whitlock, W. P., Jr. The Diophantine equation

$$A^4+2B^4=C^2+D^4.$$

Scripta Math. 17, 84–89 (1951).

Let N be a given number whose prime factors have the form $4n+1$. If N is a prime $=a^2+b^2$, two sets of forms in two parameters a, b are given for each of the four unknowns. If N is a product of 2 primes a^2+b^2, c^2+d^2, 8 sets of forms in 4 parameters a, b, c, d are given for the 4 unknowns. The author explains how to find more sets in other cases. There are misprints in lines 5, 23 on p. 85, lines 1, 6, 7 on p. 86, and lines 8–11 on p. 87. The reviewer did not perform all the extensive calculations that are necessary to check the results. *N. G. W. H. Beeger* (Amsterdam).

Moessner, Alfred. Due sistemi diofantei. Boll. Un. Mat. Ital. (3) 6, 117–118 (1951).

A two parameter solution of the Diophantine equation $\sum_{i=1}^6 x_i^3=0$ is given, and also another two parameter solution which satisfies the additional condition $\sum_{i=1}^6 x_i=0$.

I. Niven (Eugene, Ore.).

Palamà, Giuseppe. Sistemi indeterminati impossibili. Boll. Un. Mat. Ital. (3) 6, 113–117 (1951).

The set of simultaneous equations

$$x_1^k+\cdots+x_n^k=y_1^k+\cdots+y_n^k$$

($k=m, m+1, \cdots, m+n-1$) has no non-trivial real positive solutions for $m=2$, n arbitrary and for $m=3$, $n=2$. The proofs are only sketched and depend on expressing the sums of powers in terms of the elementary symmetric functions. Previously, L. Gatteschi and L. A. Rosati [Same Boll. (3) 5, 43–48 (1950); these Rev. 11, 714] have shown there are no real solutions at all for $n=m=2$. *J. W. S. Cassels.*

Wiman, A. Ein Problem bei dyadischer Zahlendarstellung. Ark. Mat. 1, 305–310 (1951).

For a prime p the expansion of $(1+x)^p$ can be written

$$\begin{aligned}(1)\quad (1+x)^p &= 1+x+x(1+x)\\ &\quad +x[1+x+x(1+x)]+x[\quad]+\cdots,\\ &= 1+x+x+x^2+x+x^2+x^2+x^3+\cdots,\end{aligned}$$

in which each square bracket contains all the previous terms of the development. Let h, k be integers with $0<h<k<p$, and let $(k, h)_p$ denote the number of times a term x^k precedes an x^h in the development (1). Since $(k, h)_p=(p-h, p-k)_p$ the restriction $h+k\leqq p$ may be introduced. It is shown that $p|(k, h)_p$ if $h+k<p$ and $p\nmid (k, h)_p$ if $h+k=p$. The method of proof is based upon writing the exponents of the terms in (1) symbolically as dyadic numbers in such a way that the resulting sequence is the increasing sequence of the first 2^p dyadic numbers. $(k, h)_p$ then becomes the number of times in this sequence that a dyadic number with k unit digits precedes one with h unit digits. *W. H. Simons.*

Davis, Alex S. The Euler-Fermat theorem for matrices. Duke Math. J. 18, 613–617 (1951).

Let $m>1$, $n>1$ be arbitrary integers, $m=\prod_{i=1}^k p_i^{e_i}$ with distinct primes p_i. Define r_i by $p_i^{r_i}\geqq n>p_i^{r_i-1}$, and w as the L.C.M. of all numbers $p_i^{r_i+e_i-1}$ and p_i^j-1 where $j=1, 2, \cdots, n$ and $i=1, 2, \cdots, k$. It is proved that w is the L.C.M. of the orders of the elements of the group G_m of non-singular matrices of order n with elements in the ring of integers (mod m). This generalizes results by J. B. Marshall [Proc. Edinburgh Math. Soc. (2) 6, 85–91 (1939); these Rev. 1, 199] and the reviewer [Duke Math. J. 15, 823–826 (1948); these Rev. 10, 183]. *I. Niven.*

Maxfield, Margaret Waugh. The order of a matrix under multiplication (modulo m). Duke Math. J. 18, 619–621 (1951).

The author determines the orders of the elements of the group G_m [see the preceding review], thus generalizing the results for the case $m=p$ of the reviewer [Duke Math. J. 15, 823–826 (1948); these Rev. 10, 183]. For $m=p^e$, the orders are $p^{e-1}u$ and their divisors as u ranges over the orders of the elements of G_p. For $m=\prod_{i=1}^k p_i^{e_i}$ the orders are L.C.M. $[v_1, \cdots, v_k]$ and their divisors as each v_i ranges over the orders of the elements of $G_{p_i^{e_i}}$. *I. Niven.*

Bateman, Paul T. On the representations of a number as the sum of three squares. Trans. Amer. Math. Soc. 71, 70–101 (1951).

The problem of representation of integers as sums of squares has received a great deal of attention [see, for in-

stance, the bibliography at the end of the paper under review]. In particular it was proved by G. H. Hardy that, for $5\leqq s\leqq 8$, the number $r_s(n)$ of representations of n as the sum of s squares is given by

$$(*)\qquad r_s(n)=\rho_s(n),$$

where

$$\rho_s(n)=\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}n^{\frac{1}{2}s-1}\mathfrak{S}_s(n)$$

and

$$\mathfrak{S}_s(n)=\sum_{k=1}^{\infty}k^{-\frac{1}{2}s}\sum_h\{\eta(h,k)\}^s e^{-\pi i h n/k};$$

here h ranges over a complete system of residues mod $2k$, whilst $\eta(h,k)$ is defined as 0 when $(h,k)>1$ and as

$$\tfrac{1}{2}k^{-\frac{1}{2}}\sum_j e^{\pi i h j^2/k}$$

when $(h,k)=1$, with j ranging over a complete system of residues mod $2k$. Hardy obtained his result by demonstrating, with the aid of the theory of modular functions, the identity of the functions

$$\Psi_s(\tau)=1+\sum_{n=1}^{\infty}\rho_s(n)e^{\pi i\tau n}$$

and

$$\{\vartheta_3(0|\tau)\}^s=1+\sum_{n=1}^{\infty}r_s(n)e^{\pi i\tau n}=\left\{\sum_{n=-\infty}^{\infty}e^{\pi i\tau n^2}\right\}^s$$

in the half-plane $\mathfrak{J}(\tau)>0$. He also noted that (*) is still valid for $s=3$ and $s=4$ but that his proof is then no longer applicable owing to difficulties arising in connexion with the convergence of series involved in the argument.

In the present paper these two exceptional cases are considered. For $s=4$ the difficulty can be overcome fairly easily by the device (due to L. J. Mordell) of restoring absolute convergence to the series in question by grouping together suitable pairs of terms. For $s=3$, however, a much more drastic modification of the argument is needed and the resulting treatment involves a considerable amount of technical detail. The author succeeds in showing that (*) continues to hold for $s=3$ by combining the ideas of Hardy and Mordell with a limiting process of a type used by E. Hecke. He begins by establishing the convergence of the "singular series" $\mathfrak{S}_3(n)$ and hence deducing the convergence, for $\mathfrak{J}(\tau)>0$, of the series defining $\Psi_3(\tau)$. It is precisely the properties of $\Psi_3(\tau)$ that are essential in the discussion of $r_3(n)$. They do not, however, appear to be amenable to direct study, and the crucial idea of the paper consists in the introduction of the auxiliary function

$$\Psi_{3,\sigma}(\tau)=1+\sum_{k=1}^{\infty}\sum_{h=-\infty}^{\infty}\frac{\{\eta(h,k)\}^3}{(hi-ki\tau)^{\frac{3}{2}}|hi-ki\tau|^{\sigma}}$$

which is shown to be defined for $\sigma>0$. The author then proves that

$$(**)\qquad \lim_{\sigma\to 0}\Psi_{3,\sigma}(\tau)=\Psi_3(\tau)$$

and thereby reduces the study of the required properties of $\Psi_3(\tau)$ to that of the corresponding properties of the more manageable function $\Psi_{3,\sigma}(\tau)$. It is now shown that

$$\Psi_{3,\sigma}(-1/\tau)=(-i\tau)^{\frac{3}{2}}|-i\tau|^{\sigma}\Psi_{3,\sigma}(\tau);$$

in view of (**) this implies

$$\Psi_3(-1/\tau)=(-i\tau)^{\frac{3}{2}}\Psi_3(\tau).$$

The limit, as $\tau\to i\infty$, of $\Psi_{3,\sigma}(1-1/\tau)$ is evaluated next and as a consequence $\lim_{\tau\to i\infty}\Psi_3(1-1/\tau)$ is obtained. From this point onward it is possible to proceed as in Hardy's treatment of the cases $s=5,\cdots,8$; and a comparison of the functions $\Psi_3(\tau)$ and $\{\vartheta_3(0|\tau)\}^3$ leads to the conclusion that they are identically equal. Hence (*) is valid for $s=3$.

By summing the singular series the author also obtains a formula for $r_3(n)$ which exhibits the fluctuations of $r_3(n)$ due to the arithmetical structure of n. We content ourselves with quoting one special case of this formula. If $n\equiv 3 \pmod 8$ and n is not divisible by the fourth power of any prime, then

$$r_3(n)=\frac{16}{\pi}n^{\frac{1}{2}}K(-4n)\prod_{p^2|n}\left\{1+\left(p-\left(\frac{-p^{-2}n}{p}\right)\right)^{-1}\right\}$$

where

$$K(-4n)=\sum_{m=1}^{\infty}\left(\frac{-4n}{m}\right)\frac{1}{m}.$$

As an incidental result of the investigation the author also obtains expressions for the number of primitive representations of n as the sum of three squares and for the number of representations of n as the sum of three or of four triangular numbers. *L. Mirsky* (Bristol).

Roth, K. F. On Waring's problem for cubes. Proc. London Math. Soc. (2) 53, 268–279 (1951).

The author proves that almost all positive integers u are representable in the form $u=p_1^3+p_2^3+p_3^3+x^3$, where p_1, p_2 and p_3 are primes and x is a positive integer, and that all large positive integers u are representable in the form $u=p_1^3+p_2^3+\cdots+p_7^3+x^3$, where $p_1,\cdots,p_7$ are primes and x is a positive integer. *L. K. Hua* (Peking).

Halberstam, H. On the representation of large numbers as sums of squares, higher powers, and primes. Proc. London Math. Soc. (2) 53, 363–380 (1951).

The author establishes asymptotic formulas for (a) $r_1(n)$, the number of representations of a large positive integer n as a sum of r squares and s primes, where $r+2s>4$; (b) $r_2(n)$, the number of representations as a sum of a kth power and two primes; and (c) $r_3(n)$, the number of representations as a sum of a kth power, two squares, and a prime. In each case the formula is of the form $R(n)\mathfrak{S}(n)$ plus an error term which contains $(\log n)^{-c}$, where c is an arbitrarily large positive number. The factor $R(n)$ takes the size of n into account and the "singular series" $\mathfrak{S}(n)$ depends only on the arithmetical properties of the integer n.

The method is the standard Hardy-Littlewood approach with modifications due to Vinogradov. It is based on the work of Estermann [Proc. London Math. Soc. (2) 42, 501–516 (1937)], where the result for $r_1(n)$ with $r=1$, $s=2$ is obtained and the problems of $r_2(n)$ and $r_3(n)$ are suggested. The proof uses finite sums and avoids the difficulty of dealing directly with the zeros of the Dirichlet L-functions by appealing to the Page-Siegel-Walfisz result on primes in an arithmetical progression [Page, Proc. London Math. Soc. (2) 39, 116–141 (1935); Walfisz, Math. Z. 40, 592–607 (1936), p. 598]. *R. D. James* (Vancouver, B. C.).

Newman, D. J. The evaluation of the constant in the formula for the number of partitions of n. Amer. J. Math. 73, 599–601 (1951).

Hardy and Ramanujan [Proc. London Math. Soc. (2) 17, 75–115 (1918)] proved using analytical methods that the number of partitions of n, $p(n)$, satisfies

$$p(n)=(1+o(1))(1/4\sqrt{3})\exp[\pi(\tfrac{2}{3}n)^{\frac{1}{2}}].$$

The reviewer [Ann. of Math. (2) 43, 437–450 (1942); these Rev. 4, 36] proved in an elementary (but complicated) way

that $p(n)=(1+o(1))an^{-1}\exp[\pi(\frac{2}{3}n)^{\frac{1}{2}}]$, but did not succeed in showing that $a=1/4\sqrt{3}$. The author proves that $a=1/4\sqrt{3}$; his proof is elementary and simple.

P. Erdös (Aberdeen).

Mikolás, Miklós. An equivalence theorem concerning Farey series. Mat. Lapok 2, 46–53 (1951). (Hungarian. Russian and English summaries)

Let $u(n)$ be the Moebius function, $\varphi(n)$ Euler's function, $M(x)=\sum_{n=1}^{x}u(n)$, $\Phi(x)=\sum_{n=1}^{x}\varphi(n)$, ρ_ν the νth Farey fraction of order ν. Littlewood proved that $M(x)=o(x^{\frac{1}{2}+\epsilon})$ is equivalent to the Riemann hypothesis and Franel proved that

$$N(x)=\sum_{\nu=1}^{\Phi(x)}\left(\rho_\nu-\frac{\nu}{\Phi(x)}\right)^2=o(x^{-1+\epsilon})$$

is equivalent to the Riemann hypothesis. The author connects these theorems by the following result: Let $g(x)$ satisfy the following conditions: $g(x)>0$, $g'(x)>0$; further, there exists an $\eta>0$ so that $g(x)/x^{\frac{1}{2}-\eta}$ decreases for x sufficiently large. Then

$$M(x)=O(x/g(x)) \quad \text{and} \quad N(x)=O(1/(g(x))^2)$$

are equivalent. Also $M(x)=o(x/g(x))$ and $N(x)=o(1/(g(x))^2)$ are equivalent. As a corollary the author obtains that $N(x)=o(1)$ is equivalent to the prime number theorem.

P. Erdös (Aberdeen).

Métral, Paul. Définitions des fonctions presque automorphes et presque θ. C. R. Acad. Sci. Paris 232, 1798–1800 (1951).

Definitions of three types of "almost automorphic" functions and forms. *J. Lehner* (Philadelphia, Pa.).

★Häusermann, Albert. Über die Berechnung singulärer Moduln bei Ludwig Schläfli. Bearbeitung der Manuskripte und Darstellung der Hauptresultate. Thesis, University of Zürich, 1943. 142 pp.

Ludwig Schläfli published only one paper on elliptic modular functions [J. Reine Angew. Math. 72, 360–369 (1870)]. However, he left a great number of manuscripts on the subject of singular moduli and class invariants, which are here listed and analyzed. Schläfli anticipated investigations by Dedekind and Weber by 10 and 20 years. It is not quite clear why he did not publish at least part of his results. Although since Schläfli's efforts the investigations of Dedekind, Klein, Fricke, Fueter, Watson and others have obtained all of his results, some of his methods and calculations seem still to be of interest. It is to be hoped that the Schläfli manuscripts on modular functions, at least in excerpts, will be included in the now appearing Gesammelte mathematische Abhandlungen [Birkhäuser, Basel, Band I, 1950; these Rev. 11, 611]. *H. Rademacher.*

Bateman, P. T., and Chowla, S. Averages of character sums. Proc. Amer. Math. Soc. 1, 781–787 (1950).

The authors consider the arithmetic means of the sums $S(n)=\sum_{m=1}^{n}\psi(m)$, where ψ is a primitive residue character modulo k. Employing an argument involving Fourier series they derive a stronger form of Paley's Ω-result [J. London Math. Soc. 7, 28–32 (1932)] that $M(\psi)=\Omega(k^{\frac{1}{2}}\log\log k)$, where $M(\psi)$ is the maximum of $|S(0)|,\cdots,|S(k-1)|$. In the case $\psi(-1)=-1$, $\alpha k\leqq n\leqq k$, where α is a number between 0 and 1, they obtain an estimate for the arithmetic mean of $S(0),\cdots,S(n-1)$, which is valuable for n not too small relative to k. Davenport [ibid. 6, 198–202 (1931)] has proved that if s is a fixed complex number with $0<\sigma=\Re(s)<1$, then for any primitive ψ we have $|L(s,\psi)|\leqq Ck^{(1-\sigma)/2}$, where $L(s,\psi)$ is Dirichlet's function and C is a constant depending on s. The authors give a very simple proof of this result in the case $\psi(-1)=1$, with a specific value of the constant. Unfortunately their method does not give Davenport's result for the case $\psi(-1)=-1$.

A. L. Whiteman (Los Angeles, Calif.).

Servera, Pedro. An elementary theorem of algebra. Revista Acad. Ci. Zaragoza (2) 5, no. 2, 75–76 (1950). (Spanish)

Let K be the cyclotomic field of the pth roots of unity (p odd, prime). The subfield of K quadratic over the rationals is generated by $\sqrt{(\pm p)}$ according as $p=\pm1 \pmod 4$, because the discriminant of the cyclotomic polynomial Φ_p is $\pm p$ times a square. *D. Zelinsky* (Evanston, Ill.).

Whiteman, Albert Leon. Finite Fourier series and cyclotomy. Proc. Nat. Acad. Sci. U. S. A. 37, 373–378 (1951).

By means of finite Fourier series the author proves generalizations of some formulae of Vandiver. Let n, s be positive integers; p an odd prime; g a generator of the cyclic group of the non-zero elements of the finite field of order p^n; $m_1, m_2, \cdots, m_s$ divisors of p^n-1; $(j_1, j_2, \cdots, j_s)$ the number of solutions in $\gamma_1, \gamma_2, \cdots, \gamma_s$ of the equation

$$1+g^{j_1+m_1\gamma_1}+g^{j_2+m_2\gamma_2}+\cdots+g^{j_s+m_s\gamma_s}=0.$$

The author evaluates the sum

$$\sum(j_1, j_2, \cdots, j_s)(j_1+k_1, j_2+k_2, \cdots, j_s+k_s),$$

where the k_i are given integers mod m_i ($i=1, 2, \cdots, s$) and the j_i range independently over $0, 1, 2, \cdots, m_i-1$. The case $s=2$ yields results of Vandiver [same Proc. 33, 236–242 (1947); 34, 62–66 (1948); 35, 681–685 (1949); these Rev. 9, 9, 412; 11, 329]. *H. D. Kloosterman* (Leiden).

Hasse, Helmut. Zur Arbeit von I. R. Šafarevič über das allgemeine Reziprozitätsgesetz. Math. Nachr. 5, 301–327 (1951).

This paper contains a detailed presentation of the ideas of Šafarevič on the general reciprocity law [cf. I. R. Šafarevič, Mat. Sbornik N.S. 26(68), 113–146 (1950); these Rev. 11, 230]. Previously Šafarevič's work was available only in Russian. *W. H. Mills* (New Haven, Conn.).

Mordell, L. J. The reciprocity formula for Dedekind sums. Amer. J. Math. 73, 593–598 (1951).

Let p and q be positive integers without a common divisor, and f an arbitrary polynomial. Methods are developed for evaluating $T=\sum f(qx+py)$, where the summation is extended over the lattice points (x, y) lying in the region K defined by $0<x<p$, $0<y<q$, $qx+py<pq$. A detailed discussion is given of the special cases $f(\xi)=\xi$ and $f(\xi)=\xi^2$. This leads to a new proof of Dedekind's result:

$$q\sum_{x=1}^{p-1}x[qx/p]+p\sum_{y=1}^{q-1}y[py/q]=\tfrac{1}{12}(p-1)(q-1)(8pq-p-q-1).$$

W. H. Mills (New Haven, Conn.).

★Artin, E. Questions de base minimale dans la théorie des nombres algébriques. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 19–20. Centre National de la Recherche Scientifique, Paris, 1950.

Let o be a Dedekind ring, F its field of quotients, E a finite algebraic extension field of F. Denote by Q the subring

of E consisting of the elements which are integral over o. It is well known that Q is a Dedekind ring, that E is the field of quotients of Q, and that, if E/F is separable, Q is finitely generated as a o-module. It is known that the last property does not always obtain in the non-separable case. The author gives the following simple example for this: Let k be a perfect field of characteristic 2, and let y be a formal power series in one variable, x, with coefficients in k which is transcendental over the field $k(x)$ of rational functions of x and all whose terms are squares. Let $F=k(x, y)$, and let o be the subring of F which consists of the integral power series in F. Then it is shown that if $E=F(\sqrt{y})$ the corresponding Q is not finitely generated as an o-module.

In the rest of the paper a very concise exposition of the Dedekind-Steinitz-Schur-Chevalley elementary divisor theory for Dedekind rings is sketched so as to yield the structure theorem for finitely generated regular o-modules. If E/F is separable, this can be applied to the (fractional) Q-ideals of E. If $\mathfrak{A}$ is such an ideal, one obtains a representation $\mathfrak{A}=\mathfrak{a}_1e_1\oplus\cdots\oplus\mathfrak{a}_ne_n$, where the $\mathfrak{a}_i$ are fractional ideals of F relative to o, where $n=[E:F]$, and where the e_i are F-linearly independent elements of E. By modifying the e_i one can vary the $\mathfrak{a}_i$'s arbitrarily, subject only to the condition that the class modulo principal ideals of their product remain fixed. The ideal $(\mathfrak{a}_1\cdots\mathfrak{a}_n)^2|e_i^{(j)}|^2$, where the $e_i^{(j)}$ are the conjugates of the e_i relative to F, is then independent of the choice of the e_i and is called the discriminant $D(\mathfrak{A})$ of $\mathfrak{A}$. By localization one proves that $D(\mathfrak{A})=N_{E/F}(\mathfrak{A})^2D(Q)$, whence $\mathfrak{a}_1\cdots\mathfrak{a}_n=N_{E/F}(\mathfrak{A})\sqrt{D(\mathfrak{A})}|e_i^{(j)}|^{-2}$. In particular, one sees from this that a necessary and sufficient condition for the existence of a minimal basis for Q over o (i.e., for Q to be a free o-module of rank n) is that the o-ideal $\sqrt{D(Q)\Delta^{-1}}$ of F be a principal ideal, where Δ is the discriminant of a defining equation for E/F. *G. Hochschild.*

van der Sluis, A. An arithmetical theorem on systems of linear differential equations. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 252–255 (1951).

Let $f_j(z)$, $j=1, 2, \cdots, n$, be entire functions, not all identically zero, which satisfy $f_j'=\sum_{k=1}^{n}c_{jk}f_k$, the coefficients being algebraic numbers with determinant $|c_{jk}|\neq 0$. Then for every algebraic number α, with one possible exception, at least one of the $f_j(\alpha)$ is transcendental. This is an extension of a theorem of R. Rado [J. London Math. Soc. 23, 267–271 (1948); these Rev. 10, 354]. *I. Niven.*

Whitworth, J. V. The critical lattices of the double cone. Proc. London Math. Soc. (2) 53, 422–443 (1951).

Let K be the convex body obtained by rotating a square of side $\sqrt{2}$ about one of its diagonals. The object of this paper is to determine the critical lattices of K, i.e. the lattices of minimum determinant whose only point interior to K is (0, 0, 0) which is at the centre of K. These lattices, of determinant $\frac{1}{2}\sqrt{6}$, are shown to have seven points on each of the two component cones of K. Furthermore, any two of the lattices may be rotated into each other. The author starts from the three possible types of critical lattices for convex bodies in 3-space determined by Minkowski [Gesammelte Abhandlungen, vol. II, Teubner, Leipzig and Berlin, 1911, pp. 1–42]. The points of a critical lattice on one of the component cones of K are shown to belong to one of five different configurations. Four of these configurations are shown to lead either to incompatible relations or to lattices whose determinant exceeds $\frac{1}{2}\sqrt{6}$. The fifth configuration leads to the critical lattice which is completely determined, except for a rotation, by the Minkowski relations when the convex body is specialized to K. *D. Derry.*

Bambah, R. P. On the geometry of numbers of non-convex star-regions with hexagonal symmetry. Philos. Trans. Roy. Soc. London. Ser. A. 243, 431–462 (1951).

Let R be a plane star region, symmetric with respect to each of the three diagonals l_1Ol_4, l_2Ol_5, l_3Ol_6 of a regular hexagon and also to each of the bisectors of the angles l_1Ol_2, l_2Ol_3, l_3Ol_4, for which the boundary points are finite except possibly those on the hexagon diagonals, and such that each of the six sets defined as the points within an angular region l_1Ol_2, l_2Ol_3, etc., complementary to R is convex. Let T_1, T_2 be the boundary points of R for which OT_1, OT_2 bisect the angles l_2Ol_1, l_1Ol_6 respectively. Then if the points of R within the angular region T_1OT_2 are included within the parallelogram with sides OT_1, OT_2 the author shows easily that the single critical lattice of R is generated by this parallelogram. Such regions R are said to be of type I. For regions R other than of type I a unique parallelogram $OA_1A_2(A_1+A_2)$ is shown to exist for which A_1, A_2, A_1+A_2 are boundary points of R such that $\angle A_1OA_2=60°$ and A_1 is within the angular region bounded by Ol_1 and the bisector of the angle l_2Ol_1. Let $\mathfrak{L}_1$ be the lattice generated by OA_1 and OA_2, $\mathfrak{L}_2$ the reflection of $\mathfrak{L}_1$ about the bisector of the angle l_2Ol_1 and Δ the determinant of these lattices. These star regions are now subdivided into types II, III, and IV according to the manner in which the boundary of R cuts certain lines which join points of $\mathfrak{L}_1$ and $\mathfrak{L}_2$. The principal result of the paper shows that, for regions R of types II and III, $\Delta(R)\geqq\Delta$ with equality if and only if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are admissible, in which case these are the only critical lattices, and that for regions of type IV a somewhat similar result holds but which is conditioned by the fact that at least one of an infinite set of parallelograms be not less than Δ. The proofs of these results, while of elementary character, involve considerable geometric detail. As an easy consequence of his results for type II regions, the author obtains the critical lattices for the region $|f(x, y)|\leqq 1$, where $f(x, y)$ is a binary cubic form with positive discriminant, which were first determined by Mordell [Proc. London Math. Soc. (2) 48, 198–228 (1943); these Rev. 5, 172]. In addition, the results for regions of types I, II and III are applied to regions R in which the boundary arcs are (1) circular, (2) parabolic and (3) hyperbolic. The results for type IV regions are applied to star shaped dodecagons. *D. Derry.*

Bambah, R. P. Non-homogeneous binary cubic forms. Proc. Cambridge Philos. Soc. 47, 457–460 (1951).

The author has proved the following theorem in his thesis [Cambridge University, 1950]: Let

$$f(x, y)=ax^3+bx^2y+cxy^2+dy^3$$

be a cubic binary form with real coefficients and positive discriminant $D=18abcd+b^2c^2-4ac^3-4db^3-27a^2d^2$. Then, if f is reduced in the sense of Hermite and x_0, y_0 are any two real numbers, there exist integers x, y, such that

$$|f(x+x_0, y+y_0)| \leqq \max\left(|f(\tfrac{1}{2}, 0)|, |f(0, \tfrac{1}{2})|, |f(\tfrac{1}{2}, \tfrac{1}{2})|, |f(\tfrac{1}{2}, -\tfrac{1}{2})|\right)$$

and there exist forms f for which the sign "=" is necessary. The present note gives a summary of the proof. In an addendum the author states that the theorem remains true, if the form is not restricted to be reduced. *J. F. Koksma* (Amsterdam).

Davis, C. S. The minimum of a binary quartic form. II. Acta Math. 85, 183–202 (1951).

Let $f(x, y)$ be a real homogeneous binary quartic form with at least one real root. Let k be a number with the property that the region of points (x, y) for which $|f(x, y)| \leqq (k+\epsilon)\Delta^2$ always contains a point of every lattice of determinant Δ other than the origin for all positive ϵ. Let k^* denote the greatest lower bound of the numbers k. The object of this paper is to determine k^* for all $f(x, y)$ or to give (upper) estimates for it. If $f(x, y)$ has multiple roots, k^* is completely determined except for the case in which all the roots are real and two of them are distinct in which case an estimate is given. For the case where $f(x, y)$ has double roots and imaginary roots the form is shown to have a chain of successive minima which are linked with the Markoff minima of indefinite binary quadratic forms.

If $f(x, y)$ has four distinct real roots the results are first developed for the form $f_m(x, y) = x^4+6mx^2y^2+y^4$, $-1 \leqq m < -\frac{1}{3}$, and then extended to the general forms by use of invariants. Where $-\frac{1}{2} < m < -\frac{1}{3}$, a region, $f_{m'}(x, y) \leqq 1$, with appropriately chosen m' from the range $-\frac{1}{3} < m' < \frac{1}{3}$ is inscribed within the region $|f_m(x, y)| \leqq 1$. As $k^*(m')$ (the number k^* for the form $f_{m'}(x, y)$) is known from part I of this paper [Acta Math. 84, 263–298 (1951); these Rev. 12, 678] an estimate for $k^*(m)$ is given by $k^*(m) < k^*(m')$. For the range $-1 \leqq m \leqq -\frac{1}{2}$ the author proves $k^*(m) \leqq 1$ and $k^*(m) = 1$ if and only if $|f_m(x, y)| \geqq 1$ for all integral (x, y) other than $(0, 0)$. This latter condition is shown to be satisfied for a set of values m of positive measure and also not to be satisfied for a set of values m of positive measure.

For the case in which $f(x, y)$ has both real and imaginary roots, all of which are distinct, the known estimates of k^* are listed and some of them are improved by inscribing a rectangle within the region $|x^4+6mx^2y^2-y^4| \leqq 1$.

D. Derry (Vancouver, B. C.).

Störmer, Horand, und Walter, Gerhard. Verschärfung eines Satzes von Mahler über konvexe Körper in inhomogener Lage. Arch. Math. 2, 346–348 (1950).

Let $0<t<1$, $n \geqq 2$. Given a distance function $F(\mathfrak{x})$ in n-space. Suppose $F(\mathfrak{x}) \leqq 1$ defines a centrally symmetric convex body of volume J, and suppose $\mathfrak{x}=\mathfrak{o}$ is the only integral vector satisfying $F(\mathfrak{x}) \leqq 2t \cdot J^{-1/n}$. Then to every vector $\mathfrak{a}$ there is an integral vector $\mathfrak{g}$ such that

$$(1) \qquad F(\mathfrak{g}+\mathfrak{a}) < \frac{(n-1)t^{n/2}+1}{t^{n-1}J^{1/n}}.$$

The proof given by the authors is very simple. By Minkowski's theorem there are n linearly independent integral vectors $\mathfrak{y}_1, \cdots, \mathfrak{y}_n$ such that $F(\mathfrak{y}_1) \leqq F(\mathfrak{y}_2) \leqq \cdots \leqq F(\mathfrak{y}_n)$ and $\prod_1^n F(\mathfrak{y}_\nu) \leqq 2^n J^{-1}$. Thus $F(\mathfrak{y}_k)^{n-k+1} \prod_1^{k-1} F(\mathfrak{y}_\nu) \leqq 2^n J^{-1}$. Since $F(\mathfrak{y}_\nu) > 2tJ^{-1/n}$, this implies

$$(2) \qquad F(\mathfrak{y}_k) < 2t^{-(k-1)/(n-k+1)} J^{-1/n} \quad (k=1, 2, \cdots, n).$$

Let $\mathfrak{a} = \sum a_k \mathfrak{y}_k$. Choose $\mathfrak{g} = \sum g_k \mathfrak{y}_k$ such that the g_k's are integers satisfying $|g_k+a_k| \leqq \frac{1}{2}$. Then

$$F(\mathfrak{g}+\mathfrak{a}) = F(\sum (g_k+a_k)\mathfrak{y}_k) \leqq \sum |g_k+a_k| F(\mathfrak{y}_k) \leqq \tfrac{1}{2} \sum_1^n F(\mathfrak{y}_k),$$

and (2) yields (1). *P. Scherk* (Saskatoon, Sask.).

Ryde, Folke. Eine neue Art monotoner Kettenbruchentwicklungen. Ark. Mat. 1, 319–339 (1951).

For irrational θ_0, $0<\theta_0<1$, the algorithm

$$\theta_n\left(\left[\frac{a_{n+1}}{\theta_n}\right]+\theta_{n+1}\right) = a_{n+1}, \quad a_{n+1} = \left[\frac{1}{1/\theta_n - [1/\theta_n]}\right], \quad n=0, 1, 2, \cdots$$

continues indefinitely and leads to a monotone non-decreasing continued fraction, namely, $a_1/sa_1 + a_2/a_2 + a_3/a_3 + \cdots$, in which $s = [1/\theta_0]$ and the positive integers a_n are non-decreasing; moreover, the continued fraction converges to θ_0. For rational θ_0, $0<\theta_0 \leqq 1$, the algorithm terminates and yields an almost monotone non-decreasing continued fraction, namely

$$a_1/sa_1 + a_2/a_2 + \cdots + a_m/a_m + 1/1, \quad a_1 \leqq a_2 < \cdots \leqq a_{m-1} < a_m,$$

whose value is θ_0. In both cases the representation of θ_0 by such continued fractions is unique. Necessary and sufficient conditions are given in order that the expansions of the irrationals φ, ψ, in $(0, 1)$ shall coincide after deleting the initial n partial quotients from the first expansion and initial m partial quotients from the second. *W. T. Scott.*

Ryde, Folke. Sur les fractions continues monotones non-décroissantes périodiques. Ark. Mat. 1, 409–420 (1951).

A monotone non-decreasing continued fraction,

$$a_1/sa_1 + a_2/a_2 + \cdots,$$

where s, a_n are positive integers and $a_1 \leqq a_2 \leqq \cdots$, is periodic if the a_n are equal for all sufficiently large indices n. Let P, Q, R be integers without common factor and let the irreducible quadratic equation $P\theta^2+Q\theta+R=0$ have a root θ_0, $0<\theta_0<1$. The author gives necessary and sufficient conditions in order that the monotone non-decreasing continued fraction expansion of θ_0 be periodic. These conditions, which are too long to give here, involve solutions of a Pell equation determined by the original quadratic equation.

W. T. Scott (Evanston, Ill.).

Rechard, O. W. The representation of real numbers. Proc. Amer. Math. Soc. 1, 674–681 (1950).

This paper deals with Everett's representation for real numbers γ. If $f(x)$ is a real continuous strictly increasing function on $0 \leqq x \leqq p$ with $f(0)=0$, $f(p)=1$ (p an integer $\geqq 2$), then Everett's algorithm furnishes a sequence of integers $c_1, c_2, \cdots$ satisfying $0 \leqq c_i \leqq p-1$ [Bull. Amer. Math. Soc. 52, 861–869 (1946); these Rev. 8, 259; for notations etc. see this review]. Let E_p denote the class of the mentioned functions $f(x)$ and E_p^* the subclass of those $f(x)$ which give a one-one correspondence between γ and the sequence $c_1, c_2, \cdots$. The author defines $F(\gamma) = \sum_i c_i/p^i$ and proves that for any $f(x) \epsilon E_p$, the thus defined associated function $F(x)$ is continuous and nondecreasing on $0 \leqq x \leqq 1$. Conversely, if $F(x)$ denotes any continuous strictly increasing function of x on $0 \leqq x \leqq 1$ such that $F(0)=0$, $F(1)=1$, then $F(x)$ is the associated function of a unique $f(x) \epsilon E_p^*$. Further, a condition is given for $F(x)$ to be the associated function for infinitely many $f(x) \epsilon E_p - E_p^*$. With these results some problems raised by Everett are solved. [The reviewer remarks that the above theory is also contained independently in some parts of the thesis of J. W. Sanders, Distribution problems for generalised dual fractions [Thesis, Free University of Amsterdam, 1950; these Rev. 12, 679]. There is a striking similarity even in the notation and definitions between the results of both authors, although the main line of Sanders' work lies in a different direction (metric investigations).] Finally, the author considers a certain homeomorphism from E_p on E_p^* (which is not considered by Sanders): If $f(x) \epsilon E_p$, then the function $F^*(x) = f(px)$ clearly is continuous and strictly increasing, whereas $F^*(0)=0$, $F^*(1)=1$. Hence $F^*(x)$ is the associated function of a well defined $f^*(x) \epsilon E_p^*$. By putting $Tf=f^*$ the homeomorphism is established. *J. F. Koksma* (Amsterdam).

Prasad, A. V. On a theorem of Khintchine. Proc. London Math. Soc. (2) 53, 310–330 (1951).

The author proves: For any real number α there exists a real number β, such that for all integers $x>0$, y:

$$|\alpha x-y+\beta|>\delta/x, \quad \text{where} \quad \delta=3/32.$$

(Khintchine had proved that such an absolute constant $\delta>0$ exists; Fukasawa had proved that $\delta=1/457$ is such a constant; Davenport [same Proc. (2) 52, 65–80 (1950); these Rev. 12, 245; see this review for references to Khintchine and Fukasawa] had proved the same result with $\delta=1/73.9$.) The author expands α as a regular continued fraction, α being considered irrational and then he constructs β in a way which is similar to Davenport's process [loc. cit.]. The case that α is rational is treated by means of approximation of α by irrationals. *J. F. Koksma* (Amsterdam).

Kogoniya, P. G. On the set of Markov numbers. Doklady Akad. Nauk SSSR (N.S.) 78, 637–640 (1951). (Russian)

The numbers $L(\alpha)=\lim\inf|q(q\alpha-p)|$, where $q>0$, p are integers, are called the Markov numbers. It is well-known that $L(\alpha)\leqq 5^{-\frac{1}{2}}$ and that there are only a denumerable set of $L(\alpha)>\frac{1}{3}$. It is shown (I) that the set of all $L(\alpha)$ has Hausdorff dimension 1 and (II) that the set of $L(\alpha)$ in $(\frac{1}{3}-\epsilon, \frac{1}{3})$ has the power of the continuum for all $\epsilon>0$. More precisely if $N\geqq 4$ and β is any number whose partial quotients do not exceed $N-2$ a simple continued fraction argument shows that there is an α whose partial quotients have upper limit N for which $L(\alpha)=(N+2\beta)^{-1}$; and analogous but more complicated results hold for $N=2, 3$. The result (I) then follows from Jarnik's estimate for the dimension of the set of β [J. F. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 49]. *J. W. S. Cassels.*

David, Marcel. Sur trois algorithmes associés à l'algorithme de Jacobi. C. R. Acad. Sci. Paris 230, 1445–1446 (1950).

David, Marcel. Caractérisation algorithmique des irrationnelles cubiques. C. R. Acad. Sci. Paris 232, 1795–1798 (1951).

The first paper describes two more variations on Jacobi's algorithm for simultaneous approximation to two real cubic irrationalities; see Jacobi, [J. Reine Angew. Math. 69, 29–64 (1868)] and David [C. R. Acad. Sci. Paris 229, 965–967 (1949); these Rev. 11, 417]. For one of the new algorithms non-periodic sequences can occur. The second paper describes certain sequences of triplets of three-dimensional vectors which in a geometrical sense generalize Jacobi's algorithm. Every periodic sequence corresponds to a unit of the cubic field; it is asserted that to every unit correspond periodic sequences. *G. Whaples* (Bloomington, Ind.).

Rogers, C. A. The asymptotic directions of n linear forms in $n+1$ integral variables. Proc. London Math. Soc. (2) 52, 161–185 (1951).

Let $x_0, x_1, \cdots, x_n$ be the $n+1$ linear forms

$$x_i=\sum_{j=0}^{n}a_{ij}u_j \qquad (i=0, 1, \cdots, n)$$

in the $n+1$ variables $u_0, u_1, \cdots, u_n$ with real coefficients a_{ij} and determinant $\Delta\neq 0$, such that $x_1, \cdots, x_n$ are not all zero for any integral values of $u_0, u_1, \cdots, u_n$ which are not all zero. We consider the lattice of points X with coordinates $(x_0, x_1, \cdots, x_n)$ corresponding to integral values of $u_0, u_1, \cdots, u_n$. We call $\mathbf{x}$ the vector $(x_1, \cdots, x_n)$ in n-dimensional space and $|\mathbf{x}|$ its length. A direction given by a unit vector $\mathbf{v}$ will be called an asymptotic direction for the forms $x_0, \cdots, x_n$ if there exists an infinite sequence $\{X^{(r)}\}$ of distinct lattice points $X^{(r)}$ with $x_0^{(r)}>0$, such that $\mathbf{x}^{(r)}\to 0$, $\mathbf{x}^{(r)}/|\mathbf{x}^{(r)}|\to\mathbf{v}$ as $r\to\infty$. This asymptotic direction will be said to be of order $O(f(x))$, when $f(x)=o(1)$, if for at least one such sequence $\{X^{(r)}\}$ the inequality $\mathbf{x}^{(r)}=O(f(x_0^{(r)}))$ holds. It is explained by the author that without loss of generality one always may put $x_0=u_0$. Hence one may speak of the asymptotic directions for n linear forms $x_1, \cdots, x_n$ in the $n+1$ variables $u_0, u_1, \cdots, u_n$. In this paper the author investigates to what extent such asymptotic directions of various orders are arbitrary. He proves 6 theorems. Theorem 1 gives a condition (condition (a)) such that if (a) is fulfilled by the system of linear forms, its asymptotic directions of any given order can be determined in terms of the asymptotic directions of a certain set of $n-1$ linear forms in n variables. Theorem 2 states that if (a) is not fulfilled, then every direction is an asymptotic direction of order $o(1)$. Theorem 3 states that given any direction $\mathbf{v}$ there are forms $x_1, x_2, \cdots, x_n$ such that the only asymptotic directions of order $o(1)$ are $\mathbf{v}$ and $-\mathbf{v}$. The proofs of these theorems are easy; Theorem 2 is a consequence of Kronecker's theorem, while Theorem 3 depends on a simple example. The main results are Theorems 4 and 5. Theorem 4 states that the forms $x_1, \cdots, x_n$ always have an asymptotic direction $\mathbf{v}$ of order $O(x^{-1/n})$ such that $-\mathbf{v}$ is either an asymptotic direction of order $O(x^{-1/n})$ or a limit point of asymptotic directions of order $O(x^{-1/n})$. Theorem 5 states that, given any direction $\mathbf{v}$ and any positive decreasing function $f(x)$ of order $o(1)$, then there are forms $x_1, \cdots, x_n$ which do not satisfy the condition (a), but which are such that the only asymptotic directions of order $O(f(x))$ are $\mathbf{v}$ and $-\mathbf{v}$. Theorem 6 is a further elaboration of Theorem 5. For applications see the paper reviewed below. *J. F. Koksma* (Amsterdam).

Rogers, C. A. The signatures of the errors of simultaneous Diophantine approximations. Proc. London Math. Soc. (2) 52, 186–190 (1951).

In this paper five theorems are proved which are applications or restatements of the general theorems of the preceding paper for the special case that the $n+1$ forms $x_0, x_1, \cdots, x_n$ mentioned there are the forms $x_0=u_0$, $x_1=u_1-\theta_1u_0, \cdots, x_n=u_n-\theta_nu_0$, where at least one of the real numbers $\theta_1, \cdots, \theta_n$ is irrational. These theorems give important information on the set of signs of the numbers $x_1, x_2, \cdots, x_n$ in connection with the closeness of the approximation of the system $\theta_1, \cdots, \theta_n$ by the system of fractions $u_1/u_0, \cdots, u_n/u_0$. *J. F. Koksma* (Amsterdam).

✓***Pisot, Charles. Quelques résultats d'approximation diophantienne.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 57–58. Centre National de la Recherche Scientifique, Paris, 1950.

This is an expository article outlining results which have been published in greater detail in the author's previous papers [e.g., Comment. Math. Helv. 19, 153–160 (1946); these Rev. 8, 194]. *R. Salem* (Cambridge, Mass.).

Lutz, Élisabeth. Sur les approximations diophantiennes linéares P-adiques. I. Théorèmes généraux. C. R. Acad. Sci. Paris 232, 587–589 (1951).

Soient P un nombre premier, Q le corps des nombres P-adiques, $|z|_P$ pour $z\varepsilon Q$ la valeur P-adique de z (avec

$|P|_P = P^{-1}$). Soit E l'ensemble des entiers P-adiques. Considérons le système des formes

$$(1)\qquad L_j(x) = x_j + \sum_{1\leq i\leq q} a_{ij}x_{p+i} \quad (1\leqq j\leqq p;\ p+q=n;\ a_{ij}\epsilon E),$$

où $(x)=(x_1, \cdots, x_n)$ désigne un point à coordonnées entières et posons $H(x)=\max|x_h|$, $h=1, 2, \cdots, n$. Alors pour tout entier rationnel $\lambda>0$ le système

$$|L_j(x)|_P \leqq P^{-\lambda}\ (j=1, 2, \cdots, p),\quad 0<H(x)\leqq P^{\lambda p n^{-1}},$$

a une solution (x) et de plus le système

$$|L_j(x)|_P\leqq P^{-\lambda}\ (j=1, 2, \cdots, p),$$
$$P^{\lambda p}(n!)^{-1}\leqq H(x^{(1)})\cdots H(x^{(n)})\leqq P^{\lambda p}$$

a n solutions linéairement indépendantes

$$(x^k)=(x_1^{(k)}, \cdots, x_n^{(k)}).$$

Si le système (1) est non-annulable, alors le système des inégalités $|L_j(x)|^p\cdot H(x)^n\leqq 1$ $(j=1, \cdots, p)$ a une infinité de solutions $(x_1, \cdots, x_n)$ primitives. Si pour un λ et un $(x^{(0)})=(x_1^{(0)}, \cdots, x_n^{(0)})$ avec $x_k^{(0)}\epsilon E$, il n'y a pas de solution $(x)=(x_1, \cdots, x_n)$ à coordonnées entières au système

$$|L_j(x)|_P\leqq P^{-\lambda}\ (j=1, \cdots, p),\quad 0<H(x)\leqq c_1P^{\lambda p n^{-1}},$$

ni au système

$$|L_j(x+x^{(0)})|_P\leqq P^{-\lambda}\ (j=1, \cdots, p),\quad 0\leqq H(x)\leqq c_2P^{\lambda p n^{-1}},$$

on a $c_1^{n-1}c_2\leqq \frac{1}{2}n$.

Ces énoncés qui sont étroitement liés aux résultats de K. Mahler [Jber. Deutsch. Math. Verein. 44, 250–255 (1934)] sont des cas spéciaux des résultats annoncés par l'auteur sur des inégalités $f(x)\leqq c$ où $f(x)$ désigne une forme hyperconvexe, c'est-à-dire que $f\geqq 0$ pour $x\epsilon Q^n$ pendant de plus $f(tx)=|t|_P f(x)$ pour tout $t\epsilon Q$ et $f(x+y)\leqq \max(f(x), f(y))$. En outre des généralisations sont annoncées au cas où l'on considère simultanément plusieurs nombres premiers $P_1, \cdots, P_l$, comme l'a fait K. Mahler dans le cas des formes linéaires. *J. F. Koksma* (Amsterdam).

Lutz, Élisabeth. Sur les approximations diophantiennes linéaires P-adiques. II. Existence de systèmes remarquables. C. R. Acad. Sci. Paris **232**, 667–669 (1951).

En utilisant la notation de la note analysée ci-dessus, l'auteur considère des systèmes (1); elle démontre l'existence de systèmes (1) spéciaux comme elle l'a fait pour le cas réel dans une note antérieure [Chabauty et Lutz, même C. R. 231, 887–888 (1950); ces Rev. 12, 483]. Quelques uns de ses résultats sont liés à ceux de K. Mahler [Mathematica, Zutphen. B. 7, 2–6 (1938), p. 5]. *J. F. Koksma.*

Lutz, Élisabeth. Sur les approximations diophantiennes linéaires P-adiques. III. Problème non homogène. C. R. Acad. Sci. Paris **232**, 784–786 (1951).

Un théorème P-adique analogue au théorème de Kronecker et quelques théorèmes qui s'y rattachent sont annoncés. L'auteur utilise la relation entre systèmes associés de formes linéaires [cf. les deux analyses ci-dessus et l'analyse ci-dessous]. *J. F. Koksma* (Amsterdam).

Lutz, Élisabeth. Sur les approximations diophantiennes linéaires P-adiques. IV. Résultats métriques. C. R. Acad. Sci. Paris **232**, 1389–1392 (1951).

Des résultats métriques sur l'approximation des formes

$$b_j + x_j + \sum_{1\leq i\leq q} a_{ij}x_{p+i} \quad (1\leqq j\leqq p;\ p+q=n)$$

qui dans le cas homogène ($b_j=0$) généralisent des résultats de Jarník [Revista Ci., Lima 47, 489–505 (1945); ces Rev. 7, 369]. La méthode est analogue à celle de Cassels dans le cas réel [Proc. Cambridge Philos. Soc. 46, 209–218 (1950); ces Rev. 12, 162]. Pour la théorie métrique P-adique cf. aussi les thèses de H. Turkstra [Amsterdam, 1936] et de D. Lock [Université libre à Amsterdam, 1947; ces Rev. 9, 79]. *J. F. Koksma* (Amsterdam).

Fel'dman, N. I. The approximation of certain transcendental numbers. II. The approximation of certain numbers connected with the Weierstrass function $\wp(z)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 153–176 (1951). (Russian)

The following results are proved: Denote by $\wp(z)$ the Weierstrass elliptic function, with algebraic invariants g_2 and g_3, further by ξ a variable algebraic number of degree n and height H, by $P(z)$ a variable polynomial $\not\equiv 0$ with integral coefficients and likewise of degree n and height H. (1) If $\omega\neq 0$ is a period of $\wp(z)$, then

$$|\xi-\omega|>\exp\{-c_1n^4(n\log n+\log H+1)\times\log^4(n\log n+\log H+2)\},$$
$$|P(\omega)|>\exp\{-c_2n^4(n\log n+\log H+1)\times\log^4(n\log n+\log H+2)\},$$

where $c_1>0$ does not depend on ξ and $c_2>0$ not on $P(z)$. (2) Let α be such that $\wp(\alpha)$ is an algebraic number, and let $\kappa>0$. Then

$$|\xi-\alpha|>\exp\{-\exp(n^{2+\kappa}+\log\log H+c_3n(n^{2+\kappa}+\log\log H)^{\frac{1}{2}})\},$$
$$|P(\alpha)|>\exp\{-\exp(n^{2+\kappa}+\log\log H+c_4n(n^{2+\kappa}+\log\log H)^{\frac{1}{2}})\},$$

where $c_3>0$ and $c_4>0$ do not depend on ξ and $P(z)$, respectively. *K. Mahler* (Manchester).

ANALYSIS

Jacobsthal, Ernst. Über das arithmetische und geometrische Mittel. Norske Vid. Selsk. Forh., Trondheim **23**, 122 (1951).

It is pointed out that the inequality $A_n\geqq G_n$ between the arithmetic and geometric means of n positive numbers follows immediately by induction from the identity

$$A_n=\frac{G_{n-1}}{n}\left[(n-1)\frac{A_{n-1}}{G_{n-1}}+\left(\frac{G_n}{G_{n-1}}\right)^n\right]$$

and Bernoulli's inequality $n-1+z^n\geqq nz$, $z\geqq 0$, with $z=G_n/G_{n-1}$. *E. F. Beckenbach* (Princeton, N. J.).

Biernacki, Mieczysław. Sur le 2 théorème de la moyenne et sur l'inégalité de Tschebycheff. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **4**, 123–130 (1950). (French. Polish Summary)

The author proves that under suitable hypotheses of integrability and continuity, and with $\phi(x)$ positive, bounded and nonincreasing, there is a ξ in $[a, b]$ such that

$$\int_a^b f(x)\phi(x)dx\Big/\int_a^b g(x)\phi(x)dx=\int_a^\xi f(x)dx\Big/\int_a^\xi g(x)dx.$$

As corollaries he deduces (1) that if $p(x)>0$, $\phi(x)$ is non-

negative and nonincreasing, and

$$\left|\int_a^t p(x)f(x)dx \Big/ \int_a^t p(x)dx\right| \leqq M, \qquad a \leqq t \leqq b,$$

then

$$\left|\int_a^b p(x)f(x)\phi(x)dx\right| \leqq M\left|\int_a^b p(x)\phi(x)dx\right|$$

[generalization of a theorem of Natanson, Doklady Akad. Nauk SSSR (N.S.) **56**, 911–913 (1947); these Rev. **9**, 136]; (2) if $p(x)$ is positive, $\phi(x)$ nonincreasing, and $f(x)$ decreasing "on the average", i.e. $\int_a^t p(x)f(x)dx/\int_a^t p(x)dx$ nonincreasing, then

$$\int_a^b p(x)dx \int_a^b p(x)f(x)\phi(x)dx \geqq \int_a^b p(x)f(x)dx \int_a^b p(x)\phi(x)dx$$

(generalization of Chebyshev's inequality).

R. P. Boas, Jr. (Evanston, Ill.).

Biernacki, M., Pidek, H., et Ryll-Nardzewski, C. Sur une inégalité entre des intégrales définies. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **4**, 1–4 (1950). (French. Polish summary)

The authors give a simple proof of the known result that for $f(x_1, \cdots, x_n), g(x_1, \cdots, x_n) \geqq 0$ in a domain D of content V we have

$$\left|\frac{1}{V}\int_D fg\,dV - \frac{1}{V^2}\int_D f\,dV\int_D g\,dV\right| \leqq \tfrac{1}{4}(\sup f)(\sup g).$$

The paper was written in 1945, but a footnote contains references to subsequent pertinent literature; for these, see also J. Karamata [Acad. Serbe Sci. Publ. Inst. Math. **2**, 131–145 (1948); these Rev. **10**, 435]. *E. F. Beckenbach.*

Ryll-Nardzewski, Czesław. Sur la dérivée logarithmique des fonctions monotones. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **4**, 9–12 (1950). (French. Polish summary)

Relative to an unpublished result of Biernacki [presented at the Fifth Polish Mathematical Congress, Cracow, 1947], the author develops inequalities concerning solutions $y(x) \neq 0$ of the differential equation $y^{(n)}(x) = A(x)y(x)$ for $x \geqq x_0$, where $A(x)$ is positive, continuous, and nondecreasing, with $y^{(i)}(x_0) \geqq 0$ for $i = 0, 1, \cdots, n-1$. Seemingly a persistent misprint, adversely affecting the validity of the analysis, originated in the preparation of the manuscript. Thus the left side of the first inequality obtained,

$$(*) \qquad \frac{u_i}{u_{i-1}} \leqq \frac{n-i}{n-i-1} \qquad (i = 1, \cdots, n-2),$$

where $u_i(x) = y^{(i)}(x)/y^{(i-1)}(x)$, apparently should be replaced by u_i/u_{i+1}; and the rest of the analysis depends on (*) as written. *E. F. Beckenbach* (Princeton, N. J.).

Dias Tavares, Armando. A theorem on a real function of a real variable. Revista Cientifica **1**, no. 1, 9–11 (1950). (Portuguese)

A proof of a theorem of É. Borel and Deltheil [Probabilités et Erreurs, Armand Colin, Paris, 8th ed., 1950] to the effect that a continuous real function of a real variable having the property that $x_1 + \cdots + x_n = 0$ implies $f(x_1) + \cdots + f(x_n) = 0$ can only be of the form $f(x) = kx$, k constant. *T. A. Botts* (Charlottesville, Va.).

Hsu, L. C. On the asymptotic evaluation of a class of multiple integrals involving a parameter. Amer. J. Math. **73**, 625–634 (1951).

The author determines the asymptotic behaviour of a class of multiple integrals of the form

$$I(\lambda) = \int_R \exp f(x, \lambda) dx_1 \cdots dx_n$$

where $f(x, \lambda) = f(x_1, \cdots, x_n, \lambda)$ is a real-valued function defined in R, a simply connected finitely bounded n-dimensional domain in euclidean n-space. It is assumed that, for every large λ, $f(x, \lambda)$ attains an absolute maximum at $x(\lambda) = (x_1(\lambda), \cdots, x_n(\lambda))$, an interior point of R, and further that $\lim_{\lambda\to\infty} x(\lambda) = \xi$ is an interior point of R. Then, under suitable restrictions,

$$I(\lambda) \sim \exp[f(x(\lambda), \lambda)]((2\pi)^n/H_n[-f(\xi, \lambda)])^{\frac{1}{2}},$$

where $H_n[-f(x, \lambda)] = \det[-\partial^2 f/\partial x_i \partial x_j]_{i, j=1, \cdots, n}$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Rogosinski, W. W. On the Cesàro and Hölder series of a function. Proc. London Math. Soc. (2) **53**, 444–459 (1951).

Let f be L-integrable in $[0, 1]$. The Cesàro means of f are defined by $c_0(x) = f(x)$, $c_k(x) = kx^{-k}\int_0^x f(t)(x-t)^{k-1}dt$. The Cesàro series is $\sum_{n=0}^\infty \sum_{r=0}^n (-1)^r \binom{n}{r} c_{1+r}(x)$. If we replace the Cesàro means by Hölder means $h_0(x) = f(x)$,

$$h_k(x) = x^{-1}\int_0^x h_{k-1}(t)dt,$$

we obtain the Hölder series of $f(x)$. The author's main results are the following. The Cesàro series converges for almost all x to the sum $f(x)$; it converges to the sum $f(x-0)$ whenever f is continuous to the left at x. If, in addition, $|f(x)|^2$ is integrable near 0, then the Hölder series is Abel-summable at x if and only if the Cesàro series converges there, the sum being the same for both series. If $f(x)(\log 1/x)^\alpha$ is integrable for all $\alpha \geqq 0$, then the Hölder series of f at x is the Laguerre series of $f(xe^{-u})$ at $u = 0$. *František Wolf.*

Zygmund, A. A remark on the integral modulus of continuity. Univ. Nac. Tucumán. Revista A. **7**, 259–269 (1950).

The function $f(x)$ is of period 2π and we write

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p},$$

$$\omega_p(h) = \omega_p(h, f) = \sup_{|t| \leqq h} \|f(x+t) - f(x)\|_p,$$

$$\omega_p^*(h, f) = \sup_{0 < t \leqq h} \|f(x+t) + f(x-t) - 2f(x)\|_p.$$

By Λ^*_p we denote the class of functions f such that

$$\omega_p^*(h, f) = O(h)$$

as h tends to zero. The author proves that if $f \varepsilon \Lambda_p^*$, $1 \leqq p \leqq \infty$, then (a) $\omega_p(h) = O(h|\log h|^{1/p})$ for $1 \leqq p \leqq 2$, (b) $\omega_p(h) = O(h|\log h|^{\frac{1}{2}})$ for $2 \leqq p < \infty$, (c) $\omega_p(h) = O(h|\log h|)$ for $p = \infty$, (d) all these estimates are best possible. The proof makes use of a theorem of Littlewood and Paley on Fourier series of functions of L^p. *A. C. Offord.*

Lozinskiĭ, S. M. On the divergence at a fixed point of interpolation processes. Doklady Akad. Nauk SSSR (N.S.) **72**, 1017–1020 (1950). (Russian)

Let $M = \{x_i^{(n)}\}$, where $1 \geqq x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} \geqq -1$, be any matrix with elements in $-1 \leqq x \leqq 1$ and let $l_k^{(n)}(x)$ be a set of interpolation functions so that $l_k^{(n)}(x_i^{(n)})$ is unity

if $i=k$ and zero if $i\neq k$. Suppose $f(x)$ is continuous in $-1\leqq x\leqq 1$. We write

$$U_n(M, f, x)=\sum_{k=1}^{n} f(x_k^{(n)})l_k^{(n)}(x).$$

The following result is typical of those proved by the author. Let $\omega(u)$ be continuous and increasing for $0\leqq u<\infty$ and such that $\omega(u_1+u_2)\leqq\omega(u_1)+\omega(u_2)$ and $\omega(0)=0$. Denote by C_ω the class of all functions $f(x)$ which satisfy

$$|f(x+h)-f(x)|\leqq K\omega(h)$$

for $-1\leqq x<x+h\leqq 1$, where K is independent of f. Then corresponding to any function ω which satisfies

$$\limsup_{n\to\infty} \omega\left(\frac{1}{n}\right) \log\log n>0$$

we can find a function f of C_ω and a number x_0 such that $U_n(M, f, x_0)$ does not tend to a limit as n tends to infinity.

A. C. Offord (London).

Bohr, Harald. A remark on uniform convergence of Dirichlet series. Mat. Tidsskr. B. 1951, 1–8 (1951). (Danish)

The manuscript was received by the editors shortly before the untimely death of the author. Here an example is given of a Dirichlet series for which both the abscissa of convergence σ_0 and the abscissa of boundedness σ_B are $-\infty$ while the abscissa of uniform convergence is $\sigma_u=+\infty$. For ordinary Dirichlet series, with $\lambda_n=\log n$, $\sigma_u=\sigma_B$ as shown by the author [C. R. Acad. Sci. Paris **151**, 375–377 (1910)]. The construction is based on the polynomial

$$P_n(z)=\tfrac{1}{4}z^n\sum_{k=1}^{n}\frac{1}{k}[z^k-z^{-k}],$$

having the property $|P_n(z)|\leqq 1$ for $|z|\leqq 1$. This inequality holds also for all partial sums of the finite sum when $-1\leqq x\leqq 0$ while there are partial sums which are $O[\log n]$ for $1-\delta_n\leqq x\leqq 1$. The desired Dirichlet series is of the form

$$f(s)=\sum_{m=1}^{\infty} m^{-2}e^{-m^2}e^{-ms}R_m(s),\quad R_m(s)=P_n[-e^{-3n(s+m)/(2m)}]$$

and $n=n_m$ is a rapidly increasing sequence. The n_m's and δ_m's are so chosen that $|R_n(s)|\leqq 1$ for $\sigma\geqq -m$, so that $\sigma_B=-\infty$ and $\sigma_0=-\infty$ is easily proved. On the other hand, there exists for every m a group of exponential terms coming from $R_m(s)$ which for $s=m$ add up to more than unity, so $\sigma_u=+\infty$.

E. Hille (New Haven, Conn.).

Ivanov, V. K. The problem of the minimax of a system of linear functions. Mat. Sbornik N.S. **28(70)**, 685–706 (1951). (Russian)

Let $F_i(z)=F_i(z_1, \cdots, z_n)$ $(i=1, \cdots, m)$ be linear functions of the complex variables $z_1, \cdots, z_n$. Put $\rho(z)=\max_i|F_i(z)|$; then $E=\min_z\rho(z)$ always exists and is called the minimax of the system $F_1, \cdots, F_m$. The value of E as well as the points where $\rho(z)=E$ (minimax points) are studied. Let z^0 be a minimax point; then the subset of $\{F_i\}$ for which $F_i(z^0)=E$ is called the boundary system of z^0. Following Kolmogorov [Uspehi Matem. Nauk **3**, no. 1(23), 216–221 (1948); these Rev. **10**, 35] it is shown that for every z we have $\Re\{\overline{F_i(z^0)}F_i(z)\}\geqq E^2$ for at least one F_i in the boundary set of z^0. Any subset of a boundary set which has the above property is called normal; normal sets have the same minimax as the original system; necessary and sufficient conditions for a subset of $\{F_i\}$ to be normal are given. The author studies in detail normal sets and related questions and derives finite algorithms for evaluating E. The paper uses only elementary methods; these and the results become especially simple when the real case, instead of the complex one, is considered. Most results for the real case have been obtained by M. G. Krein [N. Ahiezer and M. Kreĭn, On Some Problems of the Theory of Moments, Harkov, 1938 (Russian), pp. 171–199] using functional analysis.

A. Dvoretzky (Jerusalem).

Johnson, N. L., and Rogers, C. A. The moment problem for unimodal distributions. Ann. Math. Statistics **22**, 433–439 (1951).

The principal theorem is that if $n=3, 5, 7, \cdots$, or $+\infty$, and if μ_r is given for $1\leqq r<n$, then there is a unimodal distribution function $F(x)$ with mode 0 and μ_r for its moments of orders $1\leqq r<n$, if and only if there is a distribution function $G(x)$ with $(r+1)\mu_r$ for its moments of orders $1\leqq r<n$. As corollaries the authors deduce that if m, M and σ are real, $\sigma>0$, then there is a unimodal distribution function with mean m, mode M and standard deviation σ if and only if $(m-M)^2\leqq 3\sigma^2$; and a similar theorem involving third and fourth moments.

R. P. Boas, Jr. (Evanston, Ill.).

Ryll-Nardzewski, C. Sur les suites et les fonctions également réparties. Studia Math. **12**, 143–144 (1951).

The paper is devoted to the proof of the following theorem. If f is a real measurable function and if either (a) the sequence $\{f(n+t)\}$ is equally distributed mod 1 for almost all t, i.e. the asymptotic frequency of the subsequence $\{n_k\}$ such that $f(n_k+t)-[f(n_k+t)]<\alpha$ is α, or (b) the sequence $\{f(nt)\}$ is equally distributed mod 1, then f is equally distributed mod 1, i.e.

$$\lim_{T\to\infty} T^{-1}\text{ meas }\{t\,|\,f(t)-[f(t)]<\alpha,\ t\epsilon(0, T)\}=\alpha.$$

František Wolf (Berkeley, Calif.).

Hartman, S. Sur une méthode d'estimation des moyennes de Weyl pour les fonctions périodiques et presque périodiques. Studia Math. **12**, 1–24 (1951).

Weyl's mean of the function f of period 1 is the limit, if it exists, $\lim_{N\to\infty}\sum_1^N f(\kappa\xi)/N$. Weyl has shown that for irrational ξ, it is equal to $M_f=\int_0^1 f(x)dx$. The author introduces the notation $G(f, \xi, N)=\sum_1^N f(\kappa\xi)-NM_f$ and investigates estimates of G under two types of hypotheses. The first concerns the character of the number ξ expressed by its "type" [J. Koksma, Diophantische Approximationen, Springer, Berlin, 1936], the second hypothesis concerns the character of the function given by an estimate of its Fourier coefficients. A sample theorem is the following. If the Fourier coefficients of f are $O(n^{-\kappa})$ and if ξ is a number of type $\mu<\kappa$, then $G(f, \xi, N)=O(1)$. The author concerns himself also with almost periodic functions. *František Wolf.*

Theory of Sets, Theory of Functions of Real Variables

Eyraud, Henri. La recurrence finie. Ann. Univ. Lyon. Sect. A. (3) **13**, 7–8 (1950).
Eyraud, Henri. La recurrence transfinie. Ann. Univ. Lyon. Sect. A. (3) **13**, 9–19 (1950).
Eyraud, Henri. Les ordinaux et les alephs des classes transfinies. Ann. Univ. Lyon. Sect. A. (3) **13**, 21–32 (1950).

The author investigates the cardinality of well-ordered sets W defined by recurrence: every element w of W (except

the first) is determined unambiguously by a subset $M(w)$ of the set of elements of W preceding w, the power of $M(w)$ being finite (finite recurrence) or not exceeding a certain transfinite cardinal number (transfinite recurrence). In the third paper he also extends to the ordinals of the number classes ω and $\omega+1$ a method which he has discussed [Ann. Univ. Lyon. Sect. A. (3) **7**, 5–13 (1944); these Rev. **7**, 512] for representing the ordinals of the second and third number classes. *F. Bagemihl* (Rochester, N. Y.).

***Eyraud, Henri. Leçons sur la théorie des ensembles, les nombres transfinis et le problème du continu. II (supplement).** Institut de Mathématiques, Lyon, 1950. ii+48 pp.

The first two chapters are revisions of chapters 4 and 5 (dealing with the continuum problem) of the author's book [Leçons sur la théorie des ensembles, les nombres transfinis et le problème du continu, Institut de Mathématiques, Lyon, 1949; these Rev. **12**, 166]. The remaining three chapters deal with the material in the papers of the preceding review. *F. Bagemihl* (Rochester, N. Y.).

Neumer, Walter. Einige Eigenschaften und Anwendungen der δ- und ϵ-Zahlen. Math. Z. **53**, 419–449; errata **54**, 388 (1951).

Ordinals of the form ω^{ω^ν} ($\nu\geqq 0$) are called δ-numbers; an ordinal ϵ such that $\epsilon=2^\epsilon$ is called an ϵ-number. These numbers play a fundamental role in the following investigations: The author defines, in the natural manner, finite "exponent chains"

$$\alpha_1^{\cdot^{\cdot^{\cdot^{\alpha_n}}}} \quad \text{(written } [\alpha_1, \cdots, \alpha_n] \text{ for convenience),}$$

as well as certain infinite ones. After obtaining some general results on chains, he considers the particular kind of chain, $\varphi_{pqr}(\alpha)=[\alpha, \cdots, \alpha, \alpha^r, \alpha^q]$, where α^r is preceded by $p-2$ α's, $1\leqq\alpha$, $1\leqq p<\omega$, $1\leqq q<\omega$, $1\leqq r<\omega$, and $\varphi_{1qr}(\alpha)=\alpha^q$, $\varphi_{2qr}(\alpha)=[\alpha^r, \alpha^q]$. Monotonic functions are discussed, and then the points λ of discontinuity of the functions $\varphi_{pqr}(\alpha)$, as well as the corresponding limiting values $\lim_{1\leqq\nu<\lambda}\varphi_{pqr}(\nu)$ are determined. These results are applied to obtain closed expressions for values of the normal functions

$$\sigma_{pqr}(\alpha)=\sum_{1\leqq\nu<\alpha}\varphi_{pqr}(\nu) \quad\text{and}\quad \pi_{pqr}(\alpha)=\prod_{1\leqq\nu<\alpha}\varphi_{pqr}(\nu).$$

F. Bagemihl (Rochester, N. Y.).

Aigner, Alexander. Der multiplikative Aufbau der Polynome in der unendlichen Ordnungszahl ω. Monatsh. Math. **55**, 157–160 (1951).

An ordinal number is said to be a prime, if it is not expressible as the product of two ordinals less than itself. The author shows how to express any ordinal less than ω^ω as a product of a finite number of primes. This, however, was already done by Cantor [Gesammelte Abhandlungen . . . , Springer, Berlin, 1932, p. 204]. *F. Bagemihl.*

Cuesta, N. Complements to the article "Ordering of infinitesimals." Revista Mat. Hisp.-Amer. (4) **10**, 157–159 (1950). (Spanish)

This note contains the following consequences of Theorem 2 of the author's article [same Revista (4) **9**, 131–140 (1949); these Rev. **11**, 585] referred to in the title (using the same notation): (1) Every ordering of $\aleph_1$ elements can be represented geometrically in the Cartesian plane by a set of curves lying in the first quadrant, asymptotic to the x-axis, and ordered by setting $c_1<''c_2$ whenever the curve c_1 is wholly below c_2 after a certain point. (2) If $\aleph=\aleph_1$, the order type $\varphi_{\omega_1}(1)$ is characterized by the following properties: (a) it has $\aleph$ elements; (b) in every decomposition, either the initial part has the character ω_1 or the remainder has the character ω_1^*. (3) If $\aleph=\aleph_1$, every maximal fully ordered subset of S has order type $\varphi_{\omega_1}(1)$. Use is made of Theorem 15 of the author's thesis [ibid. (4) **3**, 186–205, 242–268 (1943); these Rev. **5**, 231]. *F. Bagemihl.*

Finsler, Paul. Eine transfinite Folge arithmetischer Operationen. Comment. Math. Helv. **25**, 75–90 (1951).

The author defines, for every $\alpha<\omega_1$, by transfinite induction, and with the aid of iteration and the taking of limits, a single-valued function $\varphi_\alpha(\xi, \eta)$ of an ordered pair of ordinals ξ and η less than ω_1, whose values are ordinal numbers. These functions generalize the operations $\eta+\xi=\varphi_1(\xi, \eta)$, $\eta\cdot\xi=\varphi_2(\xi, \eta)$, and $\eta^\xi=\varphi_3(\xi, \eta)$. Every $\zeta<\omega_1$ ($\zeta\neq 0, \omega$) can be written in the form $\zeta=\varphi_\alpha(\xi, \eta)$, where $\xi<\zeta$ and $\eta<\zeta$; and a definite one of all possible representations of this sort is assigned to ζ as its "principal representation" (to 0 and ω are assigned the principal representations $\varphi_1(0, 0)$ and $\varphi_1(0, \omega)$, respectively). This enables the author to solve the problem of distinguished sequences [see, e.g., Bachmann, Vierteljschr. Naturforsch. Ges. Zürich **95**, 115–147 (1950); these Rev. **12**, 165] for the limit numbers belonging to a certain segment of the second number class.

F. Bagemihl (Rochester, N. Y.).

Andreoli, Giulio. Il numero quale elemento classificatore (numeri sistemali). Ricerca, Napoli **1**, no. 1, 15–22; nos. 2–3, 10–14 (1950).

The considerations in the paper are rather fragmentary. The purpose of the author is to give various means of designating the points of a set. The scheme of successive partitions and of sequences of indices is the main tool in the author's investigations. Several kinds of order (linear, cyclic, superficial, etc.) are used also to increase the number of possible representations ("numeri sistemali"). *Đ. Kurepa.*

Kozlova, Z. I. The decomposition of certain B-sets. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 279–296 (1951). (Russian)

Let I_{xy} denote the subset of the Euclidean plane consisting of all points with both co-ordinates irrational. Let H be a Borel set in I_{xy} such that every homeomorph of H in I_{xy} is of the first Baire class. Let H have order $\alpha<\Omega$, which is defined in the following way. For a subset E of the irrational numbers, let $E=E^{[0]}$, and let α be an ordinal number $<\Omega$. If $\alpha-1$ exists, then $E^{[\alpha]}$ is the set obtained from $E^{[\alpha-1]}$ by removing all compact intervals. If $\alpha-1$ does not exist, then $E^{[\alpha]}=\bigcap_{\alpha'<\alpha}E^{[\alpha']}$. The smallest number β such that $E^{[\beta]}=0$ is the subclass of E. (This construction, and the meaning of "compact" in this context, are not clear to the reviewer.) Let P_{x_0} be the set of all $(x_0, y)\epsilon I_{xy}$. The smallest number α such that all sets $H\cap P_x$ have subclass $<\alpha$ is called the order of H. Under these assumptions on H, a proof is given that H can be written as the union of a transfinite sequence of pairwise disjoint sets N_ξ, for each of which $P_x\cap N_\xi$ is compact for all x and of order α or $\alpha-1$ as $\alpha-1$ does not or does exist. *E. Hewitt* (Uppsala).

Dieudonné, Jean. Sur la convergence des suites de mesures de Radon. Anais Acad. Brasil. Ci. **23**, 21–38 (1951).

Denote by $M(E)$ the set of all Radon measures on a locally compact space E, and by $M^+(E)$ the set of all positive Radon measures on E. For μ and ν in $M(E)$ write $\mu \leqq \nu$ in case $\nu - \mu \varepsilon M^+(E)$. Corresponding to any $\mu \varepsilon M^+(E)$, denote by μ^* the induced upper integral (defined for all non-negative extended-real functions on E). In the first part of the paper, the author establishes results concerning the strong topology of Bourbaki for $M(E)$ leading to the following proposition: if $\{\mu_n\}$ is a sequence in $M^+(E)$, increasing and bounded above, and if μ is the least upper bound of $\{\mu_n\}$, then for every non-negative real function f on E of σ-compact nucleus one has $\mu_n^*(f) \to \mu^*(f)$. An example is constructed to show that in this proposition the σ-compactness restriction cannot be simply dropped.

In the second part of the paper the author studies, for the case of compact E, the interrelations of various notions of weak convergence of a sequence $\{\mu_n\}$ in $M(E)$ to an element μ of $M(E)$, each such notion being defined by requiring $\mu_n(f) \to \mu(f)$ for all the real functions on E of some given class φ. The classes considered are φ_I: continuous functions; φ_{II}: bounded functions continuous almost everywhere, relative to each of the measures $|\mu_n|$; φ_{III}: functions bounded and (upper or lower) semi-continuous; φ_{IV}: bounded Borel functions. Among other things, it is shown that convergence in any one of these modes implies convergence in each of the preceding modes, that φ_I-convergence does not imply φ_{II}-convergence, that φ_{II}-convergence does not imply φ_{III}-convergence, that φ_{III}-convergence implies φ_{IV}-convergence if E is metrizable. Among the φ_I-convergent sequences, the φ_{II}-convergent sequences and the φ_{III}-convergent sequences are each characterized.

T. A. Botts (Charlottesville, Va.).

Revuz, André. Représentation canonique par des mesures de Radon des fonctions numériques totalement croissantes sur les espaces topologiques ordonnés. C. R. Acad. Sci. Paris **232**, 1731–1733 (1951).

The author considers an ordered topological space X and puts conditions on X which enable him to consider positive functions $F(x)$ on X which are continuous from the right and totally increasing. He gives conditions which insure the existence of a positive measure μ defined for subsets of X such that one always has $\mu[C_-(x)] = F(x)$ where $C_-(x) = \{y \varepsilon X \mid y < x\}$. If X is locally compact μ is a Radon measure. This paper provides improvements on a previous work of the author [C. R. Acad. Sci. Paris **231**, 22–24 (1950); these Rev. **12**, 108]. No proofs are given. *B. Yood* (Ithaca, N. Y.).

Hadwiger, H. Remarque sur la décomposition des ensembles de même mesure en parties (respectivement) congruentes. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **40** (1947), 50–55 (1948). (French. Polish summary)

In k-dimensional euclidean space let A and B be Lebesgue measurable sets each having diameter $\leqq D < \infty$ and with $m(A) = m(B)$. Corresponding to each $\epsilon > 0$ there is a positive number n and sets $A_\nu, B_\nu, \nu = 1, 2, \cdots, n$ such that (1) $A \supset \sum_{\nu=1}^n A_\nu$, $B \supset \sum_{\nu=1}^n B_\nu$; (2) A_ν, B_ν are measurable; (3) $A_\nu A_\mu = 0$, $B_\nu B_\mu = 0$ for $\nu \neq \mu$; (4) A_ν and B_ν are congruent; (5) $m(A - \sum_{\nu=1}^n A_\nu) = m(B - \sum_{\nu=1}^n B_\nu) < \epsilon$. This approximate decomposition theorem is proved with the additional feature that a dependence of n upon ϵ is obtained by establishing sequences A_ν and B_ν with the remainder in (5) having measure less than $D^k[2^{2+k}/n]^{\frac{1}{k}}$. *J. F. Randolph.*

Eggleston, H. G. Correction to "A property of Hausdorff measure." Duke Math. J. **18**, 593 (1951).

Cf. same J. **17**, 491–498 (1950); these Rev. **12**, 486.

Ostrowski, A. M. Generalization of a theorem of Osgood to the case of continuous approximation. Proc. Amer. Math. Soc. **1**, 648–649 (1950).

The following theorem is proved. Let $f(t, x)$ be a continuous function of the point (t, x) for $a < x < b$ and $t \geqq T$, and suppose that $\lim_{t\to\infty} f(t, x) = f(x)$, where $f(x)$ is continuous in (a, b); then for any $\epsilon > 0$ there is a subinterval J of (a, b) and a number T_0 such that $|f(t, x) - f(x)| < \epsilon$ for all $x \varepsilon J$, $t \geqq T_0$. In the statement of the Lemma, $a'' < x < b''$ should be replaced by $a'' \leqq x \leqq b''$. *W. Rudin* (Cambridge, Mass.).

Haršiladze, F. I. On functions with bounded second variation. Doklady Akad. Nauk SSSR (N.S.) **79**, 201–204 (1951). (Russian)

A function $f(x)$ defined in an interval (a, b) is said to be of bounded second variation if there is a finite number M such that for every partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ we have

$$(*) \qquad \sum_{i=1}^{n} \left| f(x_i) + f(x_{i-1}) - 2f\left(\frac{x_i + x_{i-1}}{2}\right) \right| \leqq M.$$

Every function which is of bounded variation in the classical sense is also of bounded second variation. The converse is not true since, if $f(x)$ satisfies the condition (**) $|f(x+h) + f(x-h) - 2f(x)| = O(h)$, uniformly in x, then f is of bounded second variation, and there exist continuous functions satisfying condition (**) and yet nowhere differentiable [see, e.g., Zygmund, Duke Math. J. **12**, 47–76 (1945); these Rev. **7**, 60]. Similarly, one can define functions of bounded kth variation $(k = 1, 2, \cdots)$. If $f(x)$ is periodic and satisfies (*) over $(a, b) = (0, 2\pi)$, then the Fourier coefficients of f are $O(1/n)$. Hence the Fourier series of f converges at every point of continuity and the convergence is uniform over every closed interval of continuity. A few more theorems about Fourier series of such functions are indicated. [A very similar, but not identical, definition of the kth variation had been given by N. Obrechkoff [C. R. Acad. Sci. Paris **179**, 1128–1130 (1924)] who also gave applications to Fourier series. See also T. Popoviciu, Les fonctions convexes, Actualités Sci. Ind., no. 992, Hermann, Paris, 1944, especially p. 24; these Rev. **8**, 319.]

A. Zygmund (Chicago, Ill.).

Frenkel, Yanny, and Cotlar, Mischa. Non-additive majorants and minorants in the theory of the Perron-Denjoy integral. Revista Acad. Ci. Madrid **44**, 411–426 (1950). (Spanish)

A continuous (but not necessarily additive) real function-of-intervals f on the real interval I is termed a generalized Perron majorant [minorant] for the extended-real function-of-points p on I provided (1) except on a countable subset of I the relation $-\infty \neq \underline{D}f(x) \geqq p(x)$ [$+\infty \neq \bar{D}f(x) \leqq p(x)$] holds, and (2) the upper and lower Burkill integrals Φf and φf of f are continuous functions-of-intervals on I, $\underline{D}f(x)$ and $\bar{D}f(x)$ denoting (strong) lower and upper derivatives of f at x. A well known theorem of Marcinkiewicz [see, e.g., S. Saks, Theory of the integral, Warszawa-Lwów, 1937, p. 253] is generalized as follows: if the function-of-points p is measurable on I and possesses a generalized Perron majorant f and

minorant g, then p is Perron integrable over I and its indefinite Perron integral P satisfies $(\varphi g)(J) \leqq P(J) \leqq (\Phi f)(J)$ for all intervals $J \subset I$. Analogous definitions of generalized Denjoy majorant and minorant are given, and the analogous theorem for the Denjoy integral is established. A number of intermediate results are proved, and as applications of these, two criteria for Burkill integrability are deduced.

T. A. Botts (Charlottesville, Va.).

Silverman, Edward. Definitions of Lebesgue area for surfaces in metric spaces. Rivista Mat. Univ. Parma 2, 47–76 (1951).

The purpose of the paper is to give a definition for the Lebesgue area of a Fréchet surface in a metric space. The author achieves his definition by first considering a surface in a Banach space and in this case he utilizes the ordinary definition for Lebesgue area after suitably defining the area of a triangle. Then he introduces the Banach space m of bounded sequences and shows that the Kolmogoroff principle holds for surfaces in m. Finally the author defines the Lebesgue area of a Fréchet surface S in a metric space D to be the area of a surface in m obtained from S by an isometric map from the separable subset of D consisting of the points of S into the space m. The isometric map is not unique but the Kolmogoroff principle in m assures that the area of S as defined above is independent of the isometric map.

R. G. Helsel (Columbus, Ohio).

Cecconi, Jaurès. Sul teorema di Gauss-Green. Rend. Sem. Mat. Univ. Padova 20, 194–218 (1951).

Let Σ be an oriented Fréchet surface of the type of the 2-sphere and let K be a cube in 3-space which contains the points of Σ in its interior. Assuming Σ has finite Lebesgue area and occupies a point set of zero 3-dimensional measure, the author establishes the Gauss-Green formula

$$\int\!\!\int_K\!\!\int f_z(x,y,z)\cdot O(x,y,z;\Sigma)\,dxdydz = -\int\!\!\int_\Sigma f(x,y,z)\,dydz,$$

where f and f_z are continuous in K and $O(x,y,z;\Sigma)$ is the topological index of the point (x,y,z) with respect to Σ. The surface integral is taken in the sense of Weierstrass. An example is given to show that the formula need not hold if Σ occupies a set with positive measure.

R. G. Helsel.

Theory of Functions of Complex Variables

Mironov, V. T. On the zeros of Riemann's zeta-function. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 91–94 (1951). (Russian)

The author uses results of R. Lagrange [Acta Math. 64, 1–80 (1935)] on interpolation series for analytic functions to establish the following result (corrected for a misprint by the reviewer): Let $u>1$,

$$I_n=\sum_{k=1}^{n+1}(-1)^{k-1}(n+k)!/\{k!^2(n-k+1)!(ku)^q\zeta(ku)\},$$

and $\lambda(q)=\frac{1}{2}u\{1+\limsup_{n\to\infty}(\log|I_n|)/\log n\}$. Then a necessary and sufficient condition for the Riemann hypothesis is that $\lim_{q\to\infty}\lambda(q)=\frac{1}{2}$.

L. Schoenfeld (Urbana, Ill.).

de Castro Brzezicki, Antonio. On the analytic continuation of Dirichlet series. Revista Acad. Ci. Madrid 43, 359–391 (1949). (Spanish)

The author considers the set S of Dirichlet series $\sum a_n e^{-\lambda_n s}$ obeying $\limsup(\log n)/\lambda_n=0$, $\limsup\log|a_n|/\lambda_n=0$. By various choices of what are termed neighborhoods and partial neighborhoods, pseudo-topologies are defined on S. In each, he studies the nature of the set S_0 of non-continuable series, and finds it to be in general open and dense. This treatment is compared with similar results for power series [see Pólya, Acta Math. 41, 99–118 (1917); S. Rios, Publ. Inst. Mat. Univ. Nac. Litoral 6, 237–245 (1946); these Rev. 7, 514].

R. C. Buck (Madison, Wis.).

Levin, B. Ya. The general form of special operators on entire functions of finite degree. Doklady Akad. Nauk SSSR (N.S.) 79, 397–400 (1951). (Russian)

Class P consists of entire functions of exponential type (finite degree) with no roots for $y<0$ and satisfying $h(-\pi/2)\geqq h(\pi/2)$; an additive homogeneous operator carrying functions of P into functions of P is a $\mathfrak{B}$-operator; $\mathfrak{B}$-operators possess a generalization of Bernstein's theorem on derivatives [Levin, Izvestiya Akad. Nauk. SSSR. Ser. Mat. 14, 45–84 (1950); these Rev. 11, 510]. The author characterizes those $\mathfrak{B}$-operators which are continuous in the sense that $P_n(z)\to_A P_0(z)$ implies the existence of a B such that $\mathfrak{B}P_n(z)\to_B \mathfrak{B}P_0(z)$, where $P_n\to_A P_0$ means that

$$\lim_{n\to\infty}\{\sup|P_n(z)-P_0(z)|e^{-A|z|}\}=0.$$

From the Pólya representation it follows that every continuous operator is characterized by $\phi(z,u)$, the result of applying it to e^{-uz}. The author proves the following theorem: a continuous operator is a $\mathfrak{B}$-operator if and only if $\phi(z,u)$ is of exponential type in z for bounded u, belongs to P_z for $\Im u\leqq 0$ and to $P_u{}^*$ for $\Im z\leqq 0$, and $|\phi(z,u)|\geqq|\phi(\bar z,\bar u)|$ for $\Im z\leqq 0$, $\Im u\leqq 0$ [for the definition of P^* cf. Levin, same Doklady (N.S.) 78, 1085–1088 (1951); these Rev. 13, 25]. It follows that the continuous $\mathfrak{B}$-operators permutable with differentiation are of the form $\bar F(D)f(z)=\sum \bar a_k f^{(k)}(z)$ with $F(z)$ of class P^*; those operators permutable with zd/dz are also characterized.

R. P. Boas, Jr. (Evanston, Ill.).

Charzyński, Zygmunt, et Janowski, Witold. Sur l'équation générale des fonctions extrémales dans la famille des fonctions univalentes bornées. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 41–56 (1950). (French. Polish summary)

Let F represent the family of functions $f(z)=a_1z+\cdots$ univalent and satisfying $|f(z)|<1$ in $|z|<1$. Let F_T represent the subset of F for which $a_1\geqq T$, $0<T<1$. For a functional $K(f)$ defined and real in F, the authors define the notions of differentiability and differential. They then prove that each function $w=f^*(z)$ of F for which $K(f)$ attains its maximum value has the property that on $|z|=1$, there is an open arc C such that f^* is analytic on C and transforms C into an arc of $|w|=1$. They then derive a differential equation satisfied by f^* on C. They also show that the first coefficient a_1 of the expansion of all extremal functions is equal to T.

G. Springer (Evanston, Ill.).

Janowski, Witold. Le maximum d'argument des fonctions univalentes bornées. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 57–72 (1950). (French. Polish summary)

Let F_∞ represent the class of functions which are analytic and univalent in $|z|<1$ and have $f(z)=z+A_2z^2+\cdots$. Let

F_M represent the subclass of F_∞ for which $|f(z)|<M$. The author obtains precise upper bounds for $\arg f(z)/z$, first when f is in F_M and then when f is in F_∞. The proofs depend upon the method presented in the paper reviewed above.

G. Springer (Evanston, Ill.).

★Goluzin, G. M. Some questions of the theory of univalent functions. Trudy Mat. Inst. Steklov. 27, 111 pp. (1949). (Russian)

In this little book the author gives a unified presentation of certain results of his which have already appeared in print in a series of fifteen articles. The author is concerned primarily with the class of functions S: $f(z)=z+c_2z^2+\cdots$ regular and univalent in $|z|<1$, and the class of functions Σ: $F(\zeta)=\zeta+\alpha_0+\alpha_1/\zeta+\cdots$ meromorphic and univalent in $|\zeta|>1$. The objective is to find bounds for $|c_n|$, $|\alpha_n|$, $|f(z)|$, $|f'(z)|$, $|F(\zeta)|$, $|F'(\zeta)|$, and certain rational expressions in these quantities. Properties of the extremal functions are also investigated. A detailed account of his results is not necessary here since nearly all of the original papers have been covered by previous reviews. One theorem not so covered is: if $f(z)\in S$, $r=|z|<1$, then

$$|f(z)|+|f(-z)|\leqq\frac{r}{(1-r)^2}+\frac{r}{(1+r)^2},$$

$$|f'(z)|+|f'(-z)|\leqq\frac{1+r}{(1-r)^3}+\frac{1-r}{(1+r)^3}.$$

Although the author is always careful to give due credit to other workers in the field, only such results are given as are necessary to form with his own an integrated whole. One does find in this book a complete account of Loewner's classic work, the theorem of Fekete and Szegö on odd univalent functions and some selected results of Schaeffer, Spencer and Shiffer, but the book is not intended to be encyclopedic.

The material is presented in a clear style and is a valuable contribution to the literature on the subject. Its attractiveness is considerably marred by an unusually large number of misprints. Perhaps the high point is reached in Theorem 1 of §11 where the reviewer found one misprint in the statement of the theorem and thirteen misprints (or errors) in the proof. In Theorem 2 of §5 the words "odd" and "even" must be interchanged, as the example function $z/(1-z^2)$ shows. Theorem 3 of §2 is wrong since the assertion yields $1\leqq 0$ as $z\to 0$. The theorem is correct as stated in the original work [Rec. Math. [Mat. Sbornik] N.S. 12(54), 40–47 (1943); these Rev. 5, 93], the error occuring in an inadvertent change in the definition of $f_1(z)$ when this result was included in the book. The edition of the book which the reviewer received contained in the back, a list of twelve misprints, and the first entry of this list added a new misprint, instead of correcting one.

The author introduces the symbol $\{\varphi(\xi)\}_n$ for the coefficient of ξ^n in the power series of $\varphi(\xi)$ about $\xi=0$, and this notation facilitates considerably the manipulations and the statements of results.

A. W. Goodman.

Wing, G. Milton. Averages of the coefficients of schlicht functions. Proc. Amer. Math. Soc. 2, 658–661 (1951).

Let $f(z)=z+\sum_2^\infty a_nz^n$ be analytic and univalent in $|z|<1$, $S_n(k)=\sum_{j=0}^{n-1}C_{j+k-1,k-1}a_{n-j}$, and $\sigma_n(k)=|S_n(k)|/C_{n+k,k+1}$. It is proved that for $k>1$

$$\limsup_{n\to\infty}\sigma_n(k)\leqq\frac{e^{k+1}\Gamma(k+2)\Gamma(k-1)}{(k+1)^{k+1}2^{k-1}\Gamma^2(\frac12 k)}\equiv A(k),$$

and $\lim_{k\to\infty}A(k)=1$. The bound $A(k)$ may be replaced by

$$B(k)=ke^{k+1}\Gamma^{\frac12}(2k-1)/(k+1)^k2^{k+\frac12}$$

which is an improved estimate for small values of k. It is noted that if $f(z)=z/(1-z)^2$, then $\sigma_n(k)=1$ for all positive integers k and n.

A. W. Goodman (Lexington, Ky.).

Iliev, Lyubomir. On threefold symmetric univalent functions. Doklady Akad. Nauk SSSR (N.S.) 79, 9–11 (1951). (Russian)

Let $f(z)=z+\sum_{\nu=1}^\infty a_\nu z^{3\nu+1}$ be regular and univalent in $|z|<1$, then $|a_n|^2\leqq\sum_{\nu=1}^n|a_{n-\nu}|^2/(3\nu-1)$ for $n\geqq 2$. Since $|a_1|\leqq\frac23$, this yields $|a_2|<.650$, $|a_3|<.652$, $\cdots$. Further, if $n\geqq 4$, then $z+\sum_{\nu=1}^n a_\nu z^{3\nu+1}$ is univalent in $|z|<\frac12\sqrt[3]{3}$.

A. W. Goodman (Lexington, Ky.).

Biernacki, M. Sur quelques applications de la formule de Parseval. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 23–40 (1950). (French. Polish summary)

Let the functions $f(z)=\sum a_nz^n$ and $g(z)=\sum b_nz^n$ be holomorphic in the discs C_1: $|z|<R_1$ and C_2: $|z|<R_2$, respectively, and let $H(z)=\sum a_nb_nz^n$. Then $H(z)$ is holomorphic in the disc C_3: $|z|<R_1R_2$, and the relation

$$(1)\qquad 2\pi iH(x)=\int f(z)g(x/z)z^{-1}dz$$

(with the integral taken around any circle $|z|=R_1-\epsilon$, where ϵ is sufficiently small) holds for x in C_3 and implies that if $|f|<M_1$ in C_1 and $|g|<M_2$ in C_2, then (2) $|H|<M_1M_2$ in C_3. The author uses (1) to develop refinements of (2) and of other classical inequalities. For example, by choosing $b_n=\operatorname{sgn}\bar a_n$, he obtains (for $r<R_1=1$) the estimate $\mathfrak{M}(r)\leqq M(r^{\frac12})/(1-r)^{\frac12}$, where $\mathfrak{M}(r)=\sum|a_n|r^n$. If C_1 is the unit disc and $f(0)=1$ and $|\arg f(z)|<\pi/2k$, where k is an integer, then $|a_n|<2^{1/k}$ for all n. Under the same conditions, except that k now represents any positive number greater than one,

$$|1+a_1+\cdots+a_n|\leqq 4(k+1)3^{1+1/k}n^{1/k};$$

in this, the exponent $1/k$ cannot be replaced by a smaller number. If C_1 is the unit disc, then, with the notations $2\pi I(r,f)=\int_0^{2\pi}|f(re^{i\theta})|d\theta$ and $F(z)=\sum|a_n|^2z^n$, the inequality $I(r,F)\leqq I^2(r^{\frac12},f)$ holds. For the particular case where $|a_n|=1$ for all n, this gives the estimate

$$[1-\tfrac14\log(1-r^4)]^{\frac12}<I(r,f)\leqq 1/(1-r^2)^{\frac12}.$$

If $|a_n|=n^\alpha$ ($\alpha>0$, $n=0,1,\cdots$), then

$$A_1(\alpha)/(1-r)^\alpha<I(r,f)<A_2(\alpha)/(1-r)^{\alpha+\frac12},$$

where the functions A_i depend on α only; the exponents that occur in this statement are the best possible.

If $|\arg f(z)|\leqq\alpha\pi$ in C_1 and $|\arg g(z)|\leqq\beta\pi$ in C_2, with $\alpha+\beta<\frac12$, then $|\arg H(z)|\leqq(\alpha+\beta)\pi$ in C_3, and the function $H(z^p)$ is completely p-valent in C_3. The author also treats the case where $H(z)$ is obtained by the composition of more than two Taylor series, and he obtains estimates concerning the Riemann-Liouville generalized derivative of $f(z)$.

G. Piranian (Ann Arbor, Mich.).

Newns, W. F. A note on basic sets of polynomials. Duke Math. J. 18, 735–739 (1951).

The reviewer [same J. 15, 717–724 (1948); these Rev. 10, 187] stated a theorem in too strong a form. The author corrects this and proves a better result than the corrected theorem, as follows. Let $p_n(z)=z^n+\sum_{k=0}^{n-1}p_{nk}z^k$, $|p_{nk}|\leqq\beta_{n-k}$, $h(R)=\sum_{k=1}^\infty\beta_kR^{-k}$. Then if $h(R)\leqq 1$, the base $\{p_n(z)\}$ is effective in $|z|\leqq R$.

R. P. Boas, Jr. (Evanston, Ill.).

Makar, Ragy H., and Makar, Bushra H. On algebraic simple monic sets of polynomials. Proc. Amer. Math. Soc. **2**, 526–537 (1951).

The polynomials $p_n(z)$ under discussion have degree n and leading coefficient unity, and their coefficient matrix $P=(p_{ni})$ satisfies $(P-I)^m=0$, where I is the identity matrix; m is called the degree of the set $\{p_n(z)\}$. The authors investigate the order [for terminology cf. J. M. Whittaker, Sur les séries de base de polynomes quelconques, Gauthier-Villars, Paris, 1949; these Rev. **11**, 344] of such sets subject to various additional hypotheses, obtaining sharper results than would hold without the assumption that the sets are algebraic. The following results are established. If $|p_{ni}|\leqq kn^{n\lambda}$, $k\geqq 1$, then $\{p_n(z)\}^\nu$ (the set whose matrix is P^ν) has order at most $(m+\nu-1)\lambda$ for $1\leqq\nu\leqq m-1$, $2(m-1)\lambda$ for $\nu\geqq m-1$. The same result holds if $|\pi_{ni}|$ satisfies the same condition, where (π_{ni}) is the inverse of P. If the zeros of $p_n(z)$ are in $|z|\leqq kn^\lambda$, then the set is of order at most λ. If we form a linear combination S of simple monic sets of order λ, and S is simple, monic and algebraic of degree m, then S is of order at most $m\lambda$; but if S is not algebraic, it may be of infinite order.

R. P. Boas, Jr. (Evanston, Ill.).

Wilson, R. Some extensions of Piranian's theorem. Duke Math. J. **18**, 643–651 (1951).

Let $f(z)$ be regular in $|z|<|z_0|$ except for poles of multiplicities m_ν at the points z_ν $(\nu=1,2,\cdots,r;z_\nu\neq 0)$; and let the only singularity of $f(z)$ on $|z|=|z_0|$ be a transcendental singularity of algebraic-logarithmic type and of weight $[\sigma,k]$ at z_0 [H. G. Eggleston and R. Wilson, J. London Math. Soc. **24**, 291–304 (1949); these Rev. **11**, 337]. The author shows that then

$$\sigma=1+\lim_{n\to\infty}\frac{\log|D_{np}|+n\log(|z_0|\prod_1^r|z_\nu|^{m_\nu})}{\log n}$$

and

$$k=\lim_{n\to\infty}\frac{\log|D_{np}|+n\log(|z_0|\prod_1^r|z_\nu|^{m_\nu})-(\sigma-1)\log n}{\log\log n}$$

where $p=\sum_1^r m_\nu$, D_{np} denotes the determinant of order $p+1$ with the elements $\alpha_{i,j}=a_{n+i+j}$ $(i,j=0,1,\cdots,p)$, and the variable n is required to approach infinity with the omission of an appropriate sequence of zero density. He establishes analogous results for the case where z_0 is an isolated essential singularity of finite exponential order, and for the case where $f(z)$ has on $|z|=|z_0|$ several algebraic-logarithmic singularities of which at least one is transcendental.

G. Piranian (Ann Arbor, Mich.).

Dugué, Daniel. Relation entre le nombre des valeurs asymptotiques et le nombre des valeurs doubles. C. R. Acad. Sci. Paris **232**, 1734–1735 (1951).

Let $\varphi(z)$ be an analytic and uniform function in part or in the whole complex plane which takes infinitely many times one value. The author obtains the following two results. (i) If $\varphi(z)$ has no multiple values and is not of the form $(a\exp z+b)/(c\exp z+d)$, then $\varphi(z)$ has at least three asymptotic values. (ii) If $\varphi(z)$ has no asymptotic value and no multiple points of order greater than 2, then $\varphi(z)$ takes at least four distinct double values.

S. Agmon.

√★**Garnier, R. Sur la réduction des solutions du problème de Riemann.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 199–202. Centre National de la Recherche Scientifique, Paris, 1950.

The author considers the following problem of Riemann. Let A be a square matrix of order n of functions defined on a simple, closed suitably regular curve C in the complex plane; one is to find (under suitable conditions on A) matrices Φ, Ψ, such that (1) $\Phi=A\Psi$ (on C), the elements of Φ being analytic interior to C, those of Ψ analytic exterior to C except for a possible pole at ∞. It is known that one can construct a normalized solution (n.s.) Φ, Ψ of (1), that is, a solution for which $|\Phi|$, $|\Psi|$ do not vanish on C. If (Φ_0,Ψ_0) is a particular n.s., then any other n.s. (Φ,Ψ) is given by $\Phi=\Phi_0P$, $\Psi=\Psi_0P$, where P is a matrix of polynomials of constant, nonzero determinant. The author terms Ψ, Ψ_0 (also Φ, Φ_0) equivalent. The latter property is symmetric, reflexive and transitive. The purpose of the note, carried out by the author, is to define uniquely among the n.s.'s of (1) the simplest possible one, termed the "reduced" solution.

W. J. Trjitzinsky (Urbana, Ill.).

Vekua, N. P. On a boundary problem of the theory of a function of a complex variable. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **17**, 41–46 (1949). (Russian. Georgian summary)

The author introduces the following elements in the complex plane. L is a finite collection of simple, closed, suitably smooth curves, without common points, bounding a finite connected region D^+ (origin of coordinates in D^+); D^- is the complement of D^++L; on L there are given functions $\alpha_j(t)$ belonging to a Hölder class and transforming L one-to-one into itself. The problem under consideration is to find "region-wise" analytic functions $\varphi_j(z)$ of finite order at infinity, so that on L one has

$$(1)\qquad \varphi_k^+[d_k(t_0)]=\sum_1^n G_{kj}(t_0)\varphi_j^-(t_0)+g_k(t_0),\quad k=1,\cdots,n,$$

where the coefficients belong to a Hölder class on L, and det $|G_{kj}|\neq 0$ on L. It is proved that every solution of the homogeneous problem, having at infinity order not exceeding r (r assigned, suitably great) is given by $\varphi_i(z)=\gamma_1\varphi_i^1(z)+\cdots+\gamma_m\varphi_i^m(z)$, where the γ_j are arbitrary constants, the $\varphi_i^j(z)$ certain linearly independent particular solutions $(j=1,\cdots,m)$, the first n of which have the property $\lim_{z\to\infty}z^{-r}\varphi_j^k(z)=1$ for $k=j$, $=0$ for $k\neq j$, the order of the rest at infinity being less than r. This result leads easily to the general solution of the nonhomogeneous problem (1).

W. J. Trijitzinsky (Urbana, Ill.).

McFadden, J. A. Conformal mappings for certain doubly connected domains. Quart. Appl. Math. **9**, 323–329 (1951).

A typical example: $|z|<1$ cut by $-a<z<a$ and $-b<iz<b$ is represented on an annulus. Jacobian elliptic functions are used and alternative formulae developed.

A. J. Macintyre (Aberdeen).

Ullrich, Egon. Betragflächen mit ausgezeichnetem Krümmungsverhalten. Math. Z. **54**, 297–328 (1951).

The author studies the surfaces $h=|w(x+iy)|$ given by the modulus of analytic functions $w(z)$. Using the formula

$$K=\frac{|w''|^2}{(1+|w'|^2)^2}\Re\left[\frac{w'^2}{ww''}-1\right]$$

for the Gauss curvature, he is able to characterize the classes of these surfaces for which K is of one sign in terms of the class of analytic functions with positive real part. The problem is thus related to the study of Riemann surfaces which tolerate non-constant bounded analytic functions. The surfaces for which $K\equiv 0$ are given explicitly.

P. R. Garabedian (Stanford University, Calif.).

Fourès, Léonce. Fonctions analytiques admettant une fonction d'automorphie donnée. C. R. Acad. Sci. Paris 232, 1894–1895 (1951).

The following theorem is proved. Let D and D' be two simply-connected regions of the complex plane and $z'=\Phi(z)$ a bi-uniform mapping of D on D'. There exists an analytic function $F(z)$, univalent in $D+D'$, such that $F(z)=F(z')$ whenever $z'=\Phi(z)$.

J. Lehner (Philadelphia, Pa.).

Jenkins, James A. On the topological theory of functions. Canadian J. Math. 3, 276–289 (1951).

Ce mémoire est consacré à l'étude des déformations continues des fonctions méromorphes et des transformations intérieures $w=f(z)$ définies dans $|z|<1$. Il suit de près les résultats obtenus dans cette voie par Morse et Heins [Acta Math. 79, 51–103 (1947); ces Rev. 8, 507] qui sont étendus par l'auteur au cas de l'ensemble caractéristique infini. Dans la terminologie des auteurs cités ci-dessus les affixes des zéros, des pôles et des points de ramification de $f(z)$ compris dans $|z|<1$ forment l'ensemble caractéristique (α) de $f(z)$ et les déformations envisagées de la transformation, soit $w=F(z, t)$ où F est continue de (z, t) et $F(z, 0)\equiv f(z)$, sont de plus telles que $w=F(z, t)$ est transformation intérieure quel que soit t. Si les points (α) sont fixes quand t varie, la déformation est dite restreinte (restricted). Le principal résultat démontré ici prouve que—même dans le cas de (α) infini—l'identité des invariants $J_i(f, \alpha)$ de Morse et Heins, relatifs aux mêmes points α_i de (α), est nécessaire et suffisante pour que les transformations f_1 et f_2 ayant le même ensemble caractéristique (α) appartiennent à la même classe restreinte de déformations. Si f_1 et f_2 sont méromorphes on passe de l'une à l'autre par une déformation faite de fonctions méromorphes. Le résultat s'étend évidemment aux domaines simplement connexes quelconques.

S. Stoïlow (Bucarest).

Tsuji, Masatsugu. Some metrical theorems on Fuchsian groups. Kōdai Math. Sem. Rep. 1950, 89–93 (1950).

Theorems concerning the behavior of Fuchsoid groups in the neighborhood of the unit circle are established. Application is made to the theory of meromorphic functions on Riemann surfaces with null boundary. It is shown that a function meromorphic ($\neq$const.) and not of finite valence on a Riemann surface with null boundary takes all values infinitely often except for a set of capacity zero.

M. Heins (Providence, R. I.).

Combes, Jean. Sur quelques propriétés des fonctions algébroïdes. Ann. Fac. Sci. Univ. Toulouse (4) 12, 5–76 (1950).

This paper is concerned with the study of functions on an algebroid surface. The theory of normal families is treated for families of analytic functions defined in a given Riemannian domain as well as for families of functions with variable domains. For a Riemann surface of an entire algebroid function which satisfies certain conditions with respect to the distribution of the ramification points it is shown that the analytic functions on the surface have at most one exceptional value and in addition the existence of "cercles de remplissage" and Julia directions is established. The uniformization of algebroid functions with the aid of Fuchsoid functions is studied. The Landau theorem is extended to functions algebroid in a circle. In addition, the existence on an algebroid surface of functions having assigned poles or essential singularities with given principal parts is established. The methods apply to certain more general classes of surfaces. Finally, the existence of abelian integrals of the first and second kind with assigned principal modules is established for certain classes of Riemann surfaces including the algebroids. Reference is made to the fact that the paper of Behnke and Stein in which such questions are treated for arbitrary non-compact Riemann surfaces [Math. Ann. 120, 430–461 (1949); these Rev. 10, 696] appeared after the editing of the present paper.

M. Heins (Providence, R. I.).

Leonidova, L. M. The Riemann surface for the Green's function of a multiply connected region. Mat. Sbornik N.S. 28(70), 621–634 (1951). (Russian)

An expository account of the construction of the Riemann surface corresponding to the complex Green's function for a multiply-connected region of finite connectivity. The case of the annulus receives individual attention.

A. J. Lohwater (Ann Arbor, Mich.).

Mitchell, Josephine. A theorem in the geometry of matrices. Proc. Amer. Math. Soc. 2, 276–278 (1951).

Let $Z=(z_{jk})$ be a matrix of signature (m, n). Following a method due to Hua [Amer. J. Math. 66, 470–488 (1944); these Rev. 6, 124], the author shows that the positive definite quadratic form

$$Q=\sum_{j,\, p=1}^{m}\sum_{k,\, q=1}^{n}\frac{\partial^2 \log [\det (I-Z\bar{Z}')]^{-1}}{\partial z_{jk}\partial \bar{z}_{pq}}dz_{jk}d\bar{z}_{pq}$$

defines a metric in the domain D: $I-Z\bar{Z}'>0$ which is invariant with respect to the conjunctive group of order (m, n). By a comparison of this result with a result of Bergman [Sur les fonctions orthogonales de plusieurs variables complexes . . . , Mémor. Sci. Math., no. 106, Gauthier-Villars, Paris, 1947; Interscience Publishers, New York, 1941; these Rev. 11, 344; 2, 359] to the effect that the differential form

$$\sum_{\mu,\, \nu=1}^{k}\frac{\partial^2 \log K_B(z_1, \cdots, z_k; \bar{z}_1, \cdots, \bar{z}_k)}{\partial z_\mu d\bar{z}_\nu}dz_\mu d\bar{z}_\nu,$$

where K_B is the Bergman kernel function of the domain B, defines a positive definite metric in B which is pseudo-conformally invariant, the author is led to the conjecture that $V^{-1}[\det (I-Z\bar{Z}')]^{-m-n}$, $V=$volume of D, is the Bergman kernel of D. The conjecture has been verified for the case of matrices of signature (2, 2).

P. Davis.

Fourier Series and Generalizations, Integral Transforms

Stečkin, S. B. On the problem of multipliers for trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) 75, 165–168 (1950). (Russian)

Let us write $A_n\sim B_n$ if there exist two absolute constants c_1 and c_2 such that $c_1B_n\leqq A_n\leqq c_2B_n$ for all n. Let

$$t_n(x)=\tfrac{1}{2}a_0+\sum_{1}^{n}(a_k \cos kx+b_k \sin kx).$$

Suppose further that the sequence of real numbers $\{\lambda_k\}$ satisfies $\lambda_0=0$, $\Delta\lambda_k=\lambda_k-\lambda_{k-1}\geqq 0$ and $\Delta^2\lambda_k=\lambda_k-2\lambda_{k-1}+\lambda_{k-2}\leqq 0$ and consider the trigonometric polynomial

$$\bar{\tau}_n(x)=\bar{\tau}_n(x, t_n)=\sum_{k=1}^{\infty}\lambda_k(b_k \cos kx-a_k \sin kx).$$

The author shows among other things that

$$\tilde{M}_n=\sup_{\|t_n\|\leqq 1}\|\bar{\tau}_n(t_n)\|,$$

where by $\|\varphi(x)\|$ we understand $\max_x|\varphi(x)|$, satisfies $\tilde{M}_n\sim\sum_1^n\lambda_k/k$. His results are related to those of Szegö [Schr. Königsberg. Gel. Ges. 5, 59–80 (1928)] and Fejér [Acta Univ. Szeged. Sect. Sci. Math. 2, 75–86 (1925)].

A. C. Offord (London).

Bugaec, P. T. The approximation of continuous periodic functions of two variables satisfying a Lipschitz condition by interpolating trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) 79, 381–384 (1951). (Russian)

Let H be the class of functions $f(x, y)$ of period 2π with respect to each variable and satisfying the condition $|f(x+h, y+k)-f(x, y)|\leqq M|h|^\alpha+N|k|^\beta$. Let $\bar{S}_{mn}(f, x, y)$ be the interpolating trigonometric polynomial of order (m, n) coinciding with f at the points $(2i\pi/(2m+1), 2j\pi/(2n+1))$, where $i=0, 1, \cdots, 2m$; $j=0, 1, \cdots, 2n$. Then

$$\sup_{f\in H}|\bar{S}_{mn}(f, x, y)-f(x, y)| =2\pi^{-2}\log m\log n|\sin(m+\tfrac{1}{2})x\cdot\sin(n+\tfrac{1}{2})y| \times\min\{M\pi^\alpha/m^\alpha, N\pi^\beta/n^\beta\}+\rho_{mn},$$

where $\rho_{mn}=O\{(\log m+\log n)(m^{-\alpha}+n^{-\beta})\}$.

A. Zygmund (Chicago, Ill.).

Singh, U. N. The problem of uniqueness of trigonometrical series. Bull. Allahabad Univ. Math. Assoc. 14 (1942–1950), 11–18 (1950).

The paper is a review of known uniqueness theorems.

František Wolf (Berkeley, Calif.).

Ríos, Sixto. Introduction to the theory of Fourier series. Revista Acad. Ci. Madrid 41, 43–102 (1947); 42, 9–36, 227–243 (1948). (Spanish)

Expository lectures later issued as a book [Bermejo, Madrid, 1949; these Rev. 10, 603].

Burkill, J. C. Integrals and trigonometric series. Proc. London Math. Soc. (3) 1, 46–57 (1951).

The author investigates again a definition of an integral $\int_B$ which would have the property that a trigonometric series $\sum a_n \cos nx+b_n \sin nx$ which converges to a finite $f(x)$ everywhere except in an enumerable set is such that its coefficients are given by

$$a_n=\frac{1}{\pi}\int_{B\,a}^{a+2\pi}f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int_{B\,a}^{a+2\pi}f(x)\sin nx\,dx.$$

The author calls this integral an SCP-integral, a symmetric Cesàro-Perron integral. The definition is as follows. A Denjoy (special) integrable function $M(x)$ is called an SCP majorant of $f(x)$ with basis B (meas $B=b-a$, $a, b\in B$) if (i) $M(x)$ is SC-continuous for all x: $\lim_{h\to 0}(\int_x^{x+h}-\int_{x-h}^x)M(x)dx=0$, (ii) $M(x)$ is C-continuous in B: $x\in B$ implies

$$\lim_{h\to 0}\int_x^{x+h}M(x)dx=M(x),$$

(iii) $M(a)=0$, (iv) $\liminf_{h\to 0}h^{-2}(\int_x^{x+h}-\int_{x-h}^x)M(x)dx\geqq f(x)$ p.p. and $>-\infty$ except possibly at an enumerable set. From this majorant an integral is constructed in the usual way. The integral $F(x)$ is defined only for the points of B and proves to be SC-continuous for all x, C-continuous for $x\in B$ and independent of the particular B. If we apply this integral to the integration of a trigonometric series, then we can take B to be the set of convergence of

$$\sum(a_n \sin nx-b_n \cos nx)/n.$$

František Wolf (Berkeley, Calif.).

Korovkin, P. P. On the closure of systems of Čebyšev functions. Doklady Akad. Nauk SSSR (N.S.) 78, 853–855 (1951). (Russian)

A sequence of functions $\{f_n(x)\}$, $n=0, 1, \cdots$, continuous on $[a, b]$, is a T-system if every "polynomial" $\pi_n(x)=\sum_{k=0}^n a_k f_k(x)$ $(a_n\neq 0)$ has not more than n zeros on $[a, b]$. The following result is established: If $\{f_n(x)\}$ is a T-system on $[a, b]$, then there is a perfect set $F\subset[a, b]$ such that every function continuous on F can be uniformly approximated (as closely as one wishes) by polynomials $\pi_n(x)$. The set F is the closure of a judiciously chosen sequence $\{x_n\}\in[a, b]$; and the proof is made to depend on the fact that given $\epsilon>0$ and given points $x_0, \cdots, x_n\in[a, b]$, there corresponds a $\delta_n>0$ such that if $|\pi_n(x_s)|\leqq 1$ $(s=0, 1, \cdots, n)$ and if $|x-x_k|\leqq\delta_n$ $(x\in[a, b])$ for some k, then

$$|\pi_n(x)-\pi_n(x_k)|\leqq\epsilon.$$

I. M. Sheffer (State College, Pa.).

Fine, N. J. The generalized Walsh functions. Trans. Amer. Math. Soc. 69, 66–77 (1950).

Let F be the set of all formal power-series $\bar{x}=\sum x_n\zeta^n$ of ζ, where the coefficients x_n are in a finite field of two elements 0, 1. Then F is a locally compact field with its usual topology. The author first proves that the additive group of F is self-dual and defines the generalized Walsh functions by $\psi_y(x)=\chi_1(\mu(x)\cdot\mu(y))$, where χ_1 is a suitable character of the additive group of F and $\mu(x)$ is the inverse of the mapping $\lambda(\bar{x})=\lambda(\sum x_n\zeta^n)=\sum x_n 2^{-n}$ from F onto the set of all non-negative real numbers. He then shows that, if $y=n$ is an integer, $\psi_n(x)$ gives a Walsh function [cf. author's previous paper, same Trans. 65, 372–414 (1949); these Rev. 11, 352] and that such a class of functions $\{\psi_y(x)\}$ has properties similar to the system of exponential functions $\{\exp(2\pi ixy)\}$. The author proves, namely, that $\psi_y(x)$ are characterized as solutions of a suitable functional equation and that many theorems in Fourier analysis, for instance, the Riemann-Lebesgue theorem and Fourier integral theorem, are also valid for these generalized Walsh functions. Since the Lebesgue measure on the real line is transferred to a Haar measure of F by the mapping $\mu(x)$, some of these results may be also proved by using the general theory of harmonic analysis on locally compact abelian groups.

K. Iwasawa (Princeton, N. J.).

Kac, M. On a theorem of Zygmund. Proc. Cambridge Philos. Soc. 47, 475–476 (1951).

The author gives a short proof of the result that if $f(x)$ is of class $L_p(-\infty, \infty)$, where $1\leqq p<2$, then the Fourier transform of f exists for almost all λ in the sense that $\int_A^B f(x)e^{i\lambda x}dx$ has a limit as $A\to-\infty$, $B\to\infty$ [Zygmund, same Proc. 32, 321–327 (1936)]. It is sufficient to prove the convergence almost everywhere of the series $\sum\varphi_n(\lambda)$

and $\sum\psi_n(\lambda)$, where

$$\varphi_n(\lambda)=\int_{2n}^{2n+1} f(x)e^{i\lambda x}dx \quad \text{and} \quad \psi_n(\lambda)=\int_{2n+1}^{2n+2} f(x)e^{i\lambda x}dx.$$

This is accomplished by showing that $\{\varphi_n(\lambda)\}$ and $\{\psi_n(\lambda)\}$ are sequences of orthogonal functions on $-\infty<\lambda<\infty$ with respect to the measure belonging to the density $(1-\cos\lambda)/\lambda^2$ and by appealing to a theorem of Menchoff [Fund. Math. 10, 375–420 (1927), p. 405]. *P. Hartman.*

✓*Kampé de Fériet, J. Mathematical methods used in the statistical theory of turbulence: Harmonic analysis. The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., i+ii+108 pp. (1951).

This book is based on lectures intended to give a background sufficient for the theory of the spectrum of homogeneous turbulence. In accordance with this purpose, the following topics are briefly covered: functions of bounded variation; the Riemann-Stieltjes and Lebesgue integrals; Fourier transforms and Fourier-Stieltjes transforms. Considerable space is devoted to the Fourier transforms in n dimensions of functions which vanish outside bounded sets, and it is proved that a function is the Fourier transform of a function in L_2, vanishing outside a bounded set D, if and only if it is the sum of a uniformly convergent series $\sum a_n\phi_n$, where $\sum|a_n|^2<\infty$ and the ϕ_n's are the Fourier transforms of the functions of an orthonormal sequence of functions which vanish outside D, and which are complete for the class of functions of this type.

The book is necessarily sketchy because of the mass of material covered, and in view of this fact, and of language difficulties, it may sometimes not be clear to a reader unfamiliar with the subject whether a discussion is intended to be a proof or an indication of a possible one. There are a few slips. For example, on p. 12, in subtracting out the jump of a monotone function at a discontinuity, no account is taken of the value of the function at the discontinuity. On p. 13, where absolutely continuous functions are introduced, it is stated that there are monotone continuous functions "having no derivative". *J. L. Doob* (Urbana, Ill.).

Doetsch, Gustav. Charakterisierung der Laplace-Transformation durch ihr Differentiationsgesetz. Math. Nachr. 5, 219–230 (1951).

The Laplace transform $\mathfrak{L}_s(F)=\int_0^\infty e^{-st}F(t)dt=f(s)$ has the property that if F is absolutely continuous so that $F(t)=F(0)+\int_0^t F'(\tau)d\tau$, and if $\mathfrak{L}_s(F')$ converges for some complex s with $\Re s>0$, then $\mathfrak{L}_s(F)$ also converges for s and we have $\mathfrak{L}_s(F')=s\mathfrak{L}_s(F)-F(0)$. The author shows that this formula is characteristic for the Laplace transform. More precisely, if $\mathfrak{T}_s$ is for each s, $\Re s>0$, a linear continuous functional on $L_2(0,\infty)$ and if $\mathfrak{T}_s(F')=s\mathfrak{T}_s(F)-F(0)$ whenever F', F belong to $L_2(0,\infty)$ then $\mathfrak{T}_s\equiv\mathfrak{L}_s$. This result remains valid under certain extensions of the domain space $L_2(0,\infty)$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Knopp, Konrad. Über die Konvergenzabszisse des Laplace-Integrals. Math. Z. 54, 291–296 (1951).

The author shows that the abscissa of convergence σ_c of the Laplace transform $\int_0^\infty e^{-st}d\alpha(t)$ is given by the formula $\sigma_c=\limsup t^{-1}\log\int_{[t]}^t d\alpha(t)$ where $[t]$ is an integer such that $t-1\leqq[t]<t$. The corresponding classical formulas are different in the cases $\sigma_c\geqq0$ and $\sigma_c<0$. This result has also been obtained by Ugaheri [Ann. Inst. Statist. Math., Tokyo 2, 1–3 (1950); these Rev. 12, 406]. *I. I. Hirschman, Jr.*

Puig Adam, P. The Laplace transform and the mathematical treatment of physical phenomena. Revista Mat. Hisp.-Amer. (4) 11, 52–60 (1951). (Spanish)

This is an expository article. It is pointed out that in many physical situations we have $y(t)=\int_0^t K(t,u)x(u)du$, where $x(t)$ is the cause function and $y(t)$ the effect function (t is the time). If the physical process is temporally homogeneous then $K(t,u)=K(t-u)$ and $y(t)$ is the convolution of $K(t)$ and $x(t)$. The Laplace transform may now be applied.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Bourgin, D. G. Associated transforms and convolutions. Revista Ci., Lima 51, nos. 3–4, 5–46 (1949).

On donne un noyau (1) $h(x,t)$ (défini pour $0\leqq t\leqq x$) et la transformation fonctionelle correspondante (2) $\int_0^x h(x,t)f(t)dt$. On pose le problème suivant: (A) construire un autre noyau (3) $H(p,x)$ (défini pour $x\geqq0$ et un ensemble convenable de valeurs de la variable complexe p), c'est à dire une transformation fonctionelle (4) $\int_0^\infty H(p,x)f(x)dx$, de telle sorte que l'on ait

$$(5)\quad \int_0^\infty H(p,x)dx\int_0^x h(x,t)f(t)dt=p^{-1}\int_0^\infty H(p,x)f(x)dx.$$

On voit aisément que (formellement) (5) équivaut à (6) $\int_x^\infty H(p,t)h(t,x)dt=p^{-1}H(p,x)$. On appelle (3) associé à (1), (4) associée à (2). On remarquera que, du problème (A), est bien connue la solution $h(x,t)=1$, $H(p,x)=e^{-px}$. L'auteur étudie le cas particulier (7) $h(x+t)=x+t$ et il trouve que (pour une certaine classe de fonctions f et pour $\Re(p)>0$) on a (8) $H(p,x)=\exp(-\frac{1}{2}px^2)\cdot D_{\frac{1}{2}}(x\sqrt{(2p)})$, où $D_{\frac{1}{2}}$ est la fonction du cylindre parabolique d'ordre $\frac{1}{2}$. Successivement il établit deux formules d'inversion de la transformation associée (4) dans le cas particulier (8).

Un autre problème est posé: (B) le noyau (1) étant donné, on cherche de construire une famille de noyaux $h^\alpha(x,t)$, qui dépendent d'un paramètre positif α, de telle sorte que: (I) $\int_y^x h^\alpha(x,t)h^\beta(t,y)dt=\int_y^x h^\beta(x,t)h^\alpha(t,y)dt=h^{\alpha+\beta}(x,y)$; (II) pour $\alpha=n$ (entier positif) $h^n(x,t)$ coïncide avec le noyau $n$ième itéré de (1). En supposant de connaître le noyau (3) associé à (1) et en observant que (6) entraîne (9) $\int_x^\infty H(p,t)h^n(t,x)dt=p^{-n}H(p,x)$, $(n=1,2,\cdots)$, on peut chercher de satisfaire à (I), (II) en imposant que la formule (9) soit valable aussi pour $n=\alpha$ (nombre positif quelconque). Cette méthode permet à l'auteur de résoudre le problème (B) dans le cas particulier (7). Enfin l'auteur expose de compléments variés sur les questions traitées.

A. Ghizzetti (Rome).

***Tranter, C. J. Integral Transforms in Mathematical Physics.** Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York, N. Y., 1951. ix+118 pp. $1.50.

Brief introductions are given to Laplace, Fourier, Mellin and Hankel transformations as well as to finite Fourier, Hankel and Legendre transformations. Inverse transformations are presented, accompanied in most cases by formal derivations. Each transformation reduces a characteristic differential form in the original function to an algebraic form involving the transform and characteristic boundary values of the original function. Those characteristic forms and values are indicated here without much emphasis. Each transformation is used to solve some boundary value problems in the partial differential equations of physics or engineering. The author states that these examples are to serve as a guide in selecting appropriate transforms. The book also contains short chapters on numerical evaluation

of integrals and a combination of relaxation and transformation methods. *R. V. Churchill* (Ann Arbor, Mich.).

Castoldi, Luigi. Appunti sui fondamenti del metodo degli operatori funzionali. Atti Accad. Ligure **6**, 170–200 (1950).

There is a school in Italian mathematics whose aim is to develop along classical lines (that is to say without using the theory of function spaces and modern real variable theory) a sound and rigorous foundation of operational calculus. The direct utilisation of Laplace transforms is not favoured in this school. In the present paper, the author gives a rather general definition of functional operators. In the simplest case, for an analytic function $f(p)$ which is regular in a right half-plane and such that

$$F(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} p^{-1}e^{pt}f(p)dp$$

is convergent, and for a sectionally continuous function $V(t)$, the operator $f(\Delta)$ is defined by

$$f(\Delta)V(t)=\frac{d}{dt}\int_0^t V(t-u)F(u)du.$$

There is a more general definition for the case that the integral for $F(t)$ is not convergent as it stands but becomes convergent upon replacing the factor p^{-1} of the integrand by p^{-k-1}. Applications of this definition are given.

A. Erdélyi (Pasadena, Calif.).

Brazma, N. A. Operational calculus for functions depending on a matrix parameter. Latvijas PSR Zinātņu Akad. Vēstis **1949**, No. 4(21), 123–131 (1949). (Russian)

If a matrix **A** has the canonical representation

$$\mathbf{S}[\lambda_1, \cdots, \lambda_n]\mathbf{S}^{-1},$$

then $f(\mathbf{A})=\mathbf{S}[f(\lambda_1), \cdots, f(\lambda_n)]\mathbf{S}^{-1}$, and the Laplace transform of such a matrix function as $e^{\mathbf{A}t}$ can be computed from the diagonal representation. The author gives a table of some 40 transform pairs ranging from $t^{\mathbf{A}} \doteq p^{-\mathbf{A}}\Gamma(\mathbf{A}+1)$, valid when $\Re p>0$ and the real parts of all latent roots of **A** are >-1, to the operational image of $t^{\frac{1}{2}\mathbf{A}}J_{\mathbf{A}}(t^{\frac{1}{2}})$, valid under the same conditions. He also gives an application to a system of matric partial differential equations. *A. Erdélyi.*

Polynomials, Polynomial Approximations

Teodorčik, K. F. Application of the Nyquist diagram to the solution of auto-oscillation problems. Učenye Zapiski Moskov. Gos. Univ. Fizika. **134**, kniga 5, 3–16 (1949). (Russian)

Szegő, G. On certain special sets of orthogonal polynomials. Proc. Amer. Math. Soc. **1**, 731–737 (1950).

The author studies the polynomials $P_n(x; a, b)$ (defined by the case $\lambda=\frac{1}{2}$ of (*) below) introduced by F. Pollaczek [C. R. Acad. Sci. Paris **228**, 1363–1365, 1553–1556, 1998–2000 (1949); these Rev. **10**, 703; **11**, 104] and gives a new proof of the orthogonality relation based on the use of Laplace transforms. He then compares the asymptotic relations of Pollaczek's polynomials with those of the normalized Jacobi polynomials and gives the corresponding formulae. Finally he establishes the orthogonality relation of the polynomials $P_n^{(\lambda)}$ defined by

$$(*)\quad (1-ze^{i\theta})^{-\lambda+i\varphi(\theta)}(1-ze^{-i\theta})^{-\lambda-i\varphi(\theta)}=\sum_{n=0}^{\infty}P_n^{(\lambda)}(\cos\theta; a; b)z^n,$$

where $\varphi(\theta)=(a\cos\theta+b)/2\sin\theta$.

A. C. Offord (London).

Varma, R. S. On Appell polynomials. Proc. Amer. Math. Soc. **2**, 593–596 (1951).

Paralleling a result of the reviewer [Bull. Amer. Math. Soc. **51**, 739–744 (1945); these Rev. **7**, 64], it is shown that the polynomials defined by $P_n(x)=\int_0^\infty K_n(x, t)d\beta(t)$ form an Appell set. Here $\beta(t)$ is of bounded variation on $(0, \infty)$, $K_n(x, t)=\sum_{j=0}^n \delta_j(t)(x+t)^{n-j}/(n-j)!$, and the $\delta_j(t)$ are functions for which the integrals $I_{n,r}=\int_0^\infty \delta_n(t)t^r d\beta(t)$ exist $(n, r=0, 1, \cdots)$ with $I_{0,0}\neq 0$. This is illustrated by examples, one of which is $K_n(x, t)=[x^n/n!]\cdot {}_3F_2(-n, a, b; c, d; -t/x)$.

I. M. Sheffer (State College, Pa.).

Kingsley, Edward H. Bernstein polynomials for functions of two variables of class $C^{(k)}$. Proc. Amer. Math. Soc. **2**, 64–71 (1951).

Writing $\lambda_{n,p}(x)=C_{n,p}x^p(1-x)^{n-p}$, the Bernstein polynomials for a function $\varphi(x, y)$ of two variables are defined as

$$B_{m,n}(x, y)=\sum_0^n\sum_0^m \varphi(p/n, q/m)\lambda_{n,p}(x)\lambda_{m,q}(y).$$

The author proves that if all the partial derivatives of $\varphi(x, y)$ up to and including those of the kth order are continuous in a region R, then $\lim_{m,n\to\infty}B_{m,n}^{(i,k-i)}=\varphi^{(i,k-i)}$ uniformly in R, where $\varphi^{(i,k-i)}=\partial^{(k)}\varphi/\partial x^i\partial y^{k-i}$. For one variable and for $k=0$, this is of course the well known theorem of Bernstein. The general case for one variable was proved by Wigert [Ark. Mat. Astr. Fys. **22B**, no. 9 (1932)] and the author now extends Wigert's result to two variables.

A. C. Offord (London).

Timan, A. F., and Dzyadyk, V. K. On best approximation of quasi-smooth functions by ordinary polynomials. Doklady Akad. Nauk SSSR (N.S.) **75**, 499–501 (1950). (Russian)

The function $f(x)$ is defined in the interval (a, b) and has a derivative of the rth order which satisfies the relation

$$(1)\quad |f^{(r)}(x_1)-2f^{(r)}(\tfrac{1}{2}(x_1+x_2))+f^{(r)}(x_2)|\leqq M|x_1-x_2|$$

for all (x_1, x_2) in (a, b). The author shows, that when this is the case, the best approximation $E_n(f)$ to $f(x)$ by ordinary polynomials of degree n satisfies (2) $E_n(f)=O(n^{-r-1})$. Conversely, if for all n the best approximation to $f(x)$ satisfies (2), then $f(x)$ has a derivative of the rth order in (a, b) which satisfies (1) for all (x_1, x_2) lying in any interval (a_1, b_1) strictly contained in (a, b). *A. C. Offord* (London).

Special Functions

Peremans, W. An integral representation of $e^{itx\log x}$. Math. Centrum Amsterdam. Rapport ZW 1950-021, 8 pp. (1950). (Dutch)

The integral representation

$$\exp(itx\log x)=\int_0^\infty f(u)\exp(itx\log u)du$$

with x, t positive and

$$f(u)=(2\pi)^{-1}t\int_0^\infty \exp(itus\log s)ds$$

is proved in detail. *A. Erdélyi* (Pasadena, Calif.).

Lekkerkerker, C. G. Remarks on a formula for $e^{-tx\log x}$. Math. Centrum Amsterdam, Rapport ZW 1951-004, 11 pp. (1951). (Dutch)

This paper, like the one of the preceding review, was inspired by D. van Dantzig, who conjectured some of the results and posed a number of questions. The author first gives two proofs of the relation

$$2\pi i\exp(-x\log x) = \int_{-\infty}^{(0+)}\exp(-x\log w)\int_0^1\exp(-vw\log v)dvdw$$

which is valid for $\Re(x)>0$. He also obtains a similar integral representation of $\exp(-tx\log x)$, shows that the loop around the negative real axis can be replaced by a suitable contour, and that the integral representation breaks down when x approaches imaginary values. In the last section he uses integration by parts to make certain deductions from the integral representation. *A. Erdélyi.*

Heuman, Carl. Zur Theorie der elliptischen Integrale. Trans. Roy. Inst. Tech. Stockholm no. 32, 58 pp. (1950).

The elliptic integral

$$\int_0^\varphi \frac{\xi-\eta\sin^2\varphi}{1-p\sin^2\varphi}\frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}$$

can be represented on a plane by a line element of direction p at the point (ξ,η). When $\eta\neq p\xi$, the equation $(\eta-p\xi)^2\cdot R=p(p-k^2)(p-1)$ determines a number R called the characteristic of the integral. When two line elements belong to the same straight line the corresponding integrals have the same characteristic. The line elements which represent integrals of a given characteristic R therefore lie on a family of lines. If $R\neq 0$, these lines envelop a curve of the third class. The paper is concerned with an exhaustive examination of the properties of certain related and associated sets and sequences of tangents to the curve in the case $R<0$. The author states that there is a parallel development for the case $R>0$. *L. M. Milne-Thomson* (Greenwich).

Sears, D. B. Transformations of basic hypergeometric functions of any order. Proc. London Math. Soc. (2) 53, 181–191 (1951).

The author has previously published three related papers in the same Proceedings [(A) 52, 14–35 (1950); (B) 53, 138–157 (1951); and (C) 53, 158–180 (1951); these Rev. 12, 257; 13, 33]. In the present paper the results of (C) on hypergeometric series are carried over to basic hypergeometric series (so that (C) and the present paper are related to each other as (A) and (B)). *A. Erdélyi* (Pasadena, Calif.).

Ossicini, Alessandro. Immediata limitazione delle derivate di ordine superiore dei polinomi ultra-sferici. Boll. Un. Mat. Ital. (3) 6, 110–112 (1951).

Bounds for $d^sP_n^{(\lambda)}(x)/dx^s$ in the interval $(-1,1)$. [Reviewer's remark: The derivative in question is a multiple of $P_{n-s}^{(\lambda+s)}(x)$, and the maximum of this function in $(-1,1)$ is known. See G. Szegö, Orthogonal Polynomials [Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; these Rev. 1, 14], equation (4.7.14) and theorem 7.33.1]. *A. Erdélyi* (Pasadena, Calif.).

Tricomi, Francesco G. Sugli zeri dei polinomi sferici ed ultrasferici. Ann. Mat. Pura Appl. (4) 31, 93–97 (1950).

In this paper an approximate formula for the rth zero θ_r of the ultraspherical polynomial $P_n^{(\lambda)}(\cos\theta)$, $0<\lambda<1$, due to Gatteschi, is simplified. A similar simplification was given by Gatteschi in a recent paper [Bull. Un. Mat. Ital. (3) 5, 305–313 (1950); these Rev. 12, 607]. *O. Szász.*

Orts, J. M.ª. Recurrent series of Legendre polynomials. Collectanea Math. 3, 105–120 (1950). (Spanish)

This paper is concerned with series $\sum_{n=0}^\infty a_nP_n(z)$, where (a_n) is a series whose terms satisfy a linear recurrent relation and the $P_n(z)$ are Legendre polynomials. The function $F(z)$, defined by the series of polynomials $\sum a_nP_n(z)$, is uniform in its ellipse of convergence, and $\sum_{n=0}^\infty a_nP_n(z)=-\sum_{k=1}^m A_k/(1-2zz_k+z_k^2)^{\frac12}$, where the z_k are the poles of the function defined by the recurrent series of powers $\sum a_nz^n$ and the A_k are the corresponding residues. Generalizations and consequences of this result are obtained. *E. Frank* (Chicago, Ill.).

Kapica, P. L. The computation of the sums of negative even powers of roots of Bessel functions. Doklady Akad. Nauk SSSR (N.S.) 77, 561–564 (1951). (Russian)

Let $\lambda_1,\lambda_2,\cdots$ be the positive zeros of $J_\nu(z)=0$, arranged in ascending order, and put $\sigma_\nu(r)=\sum_{k=1}^\infty\lambda_k^{-2r}$. Values of this sum for $r=1(1)5$ and $r=8$ are known [G. N. Watson, Bessel functions, Cambridge Univ. Press, 1944, §15.51; these Rev. 6, 64]. The author gives explicit expressions for $r=6, 7, 9, 10$. He also gives the similar sums for roots of the equation $zJ_\nu'(z)-HJ_\nu(z)=0$ [cf. H. Lamb, Proc. London Math. Soc. (1) 15, 270–274 (1884)]. *A. Erdélyi* (Pasadena, Calif.).

Bose, N. N. On some integrals involving the hypergeometric function ${}_2F_1(a,b;c;-x)$. Math. Z. 54, 160–167 (1951).

The author uses operational methods to deduce integrals whose integrands contain functions of the hypergeometric type. Many of his results are either known or obvious consequences of known relations. *A. Erdélyi.*

Campbell, R. Comportement des fonctions de Mathieu associées pour les grandes valeurs des paramètres. Ann. Inst. Fourier Grenoble 2 (1950), 113–121 (1951).

Associated Mathieu functions are periodic solutions of the differential equation

$$d^2U/d\xi^2-2\nu\tan\xi\, dU/d\xi+(k^2\sin^2\xi+a)U=0.$$

The author assumes that ν, k^2, and also the characteristic value of a, are large and expands k^2, a, and $U(\xi)$ in descending powers of ν. *A. Erdélyi* (Pasadena, Calif.).

Ayant, Yves. Représentation graphique de l'intégrale des équations de Bloch. C. R. Acad. Sci. Paris 233, 245–247 (1951).

In certain cases the formal solution of the Bloch equations can be written as

$$f(t)=e^{-i\varphi(t)}i\omega_1M_z\int_{-\infty}^t h(t-u)dz(u)$$

where $z(u)=\int^u e^{-i\varphi(v)}dv$, $\varphi(v)=\int^v\Delta\omega(\tau)d\tau$. [There is a mis-

print in the definition of s.] The author discusses, by graphical methods, $f(t)$ when $\Delta\omega(\tau)=\Delta\omega_0 \sin(2\pi\tau/T)$.
A. Erdélyi (Pasadena, Calif.).

Kopineck, Hermann-Josef. Zweizentrenintegrale mit 2s- und 2p-Funktionen. II. Ionenintegrale. Z. Naturforschung **6a**, 177–183 (1951).

[Part I appeared in the same Z. **5a**, 420–431 (1950); these Rev. **12**, 410.] In the second part there are four and a half pages of formulas and one page of numerical values.
A. Erdélyi (Pasadena, Calif.).

Lundqvist, Stig O., and Löwdin, Per-Olov. On the calculation of certain integrals occurring in the theory of molecules, especially three-centre and four-centre integrals. Ark. Fys. **3**, 147–154 (1951).

By using a method based on the expansion of the mutual distance of two variable points in spherical harmonics, the authors deduce a general formula for potential integrals, and apply it to integrals of atomic *S*-functions. The formulas obtained remain useful when the wave functions are given numerically rather than by an analytic expression.
A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Myrberg, Lauri. Über reguläre und irreguläre Randpunkte des harmonischen Masses. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **91**, 12 pp. (1951).

The author considers a plane region whose boundary has positive capacity. A point of the transfinite kernel of the boundary is termed regular provided that the Green's function has zero limit at the point; otherwise a point of the transfinite kernel is termed irregular. Sufficient conditions are given for regularity and for irregularity in certain special cases. The results are related to those of G. af Hällström [Acta Soc. Sci. Fennicae. Nova Ser. A. **3**, no. 5 (1944); these Rev. **7**, 447]. The method employed consists in the use of elementary comparison functions which are easily controlled in terms of the geometric data.
M. Heins (Providence, R. I.).

Korovkin, P. P. On the growth of functions. Doklady Akad. Nauk SSSR (N.S.) **78**, 1081–1084 (1951). (Russian)

The results concern functions that are analytic in certain regions. Let D be a region in the complex plane whose boundary Γ is of positive capacity and is assumed to be a bounded set. It is shown that if (i) $f(z)$ is single-valued and regular inside D except for poles at the points $z_i \varepsilon D$, $i=1, \cdots, n$, of order m_i, (ii) $\limsup_{z\to x}|f(z)| \leqq F(x)$, $z\varepsilon D$, $x\varepsilon\Gamma$, where $F(x)$ is a continuous function, different from zero, then $|f(z)| \leqq \exp\{\sum_{i=1}^n m_i g(z, z_i)+v_{\Gamma,\varphi}(z)\}$, where $g(z, z_i)$ is the Green's function for D with singularity at point z_i, and $v_{\Gamma,\varphi}(z)$ is the solution of the Dirichlet problem in region D with boundary values $\varphi(x)=\ln F(x)$. The proof uses the property of the subharmonic function

$$u(z)=\ln|f(z)|-\sum_{i=1}^n m_i g(z, z_i)-v_{\Gamma,\varphi}(z)$$

that $\limsup_{z\to x} u(z)=0$ (x a regular point of Γ), so $u(z)\leqq 0$ in D.

Let D be a region containing the point $z=\infty$, and F its complement; let $E\subset F$ be a set of type F_σ, and let F_n be a closed set ($F_n\subset E$) and D_n the complement to F_n. Let $\varphi(x)$ be continuous on F and $v_n(z)$ the solution of the Dirichlet problem for region D_n with boundary values $\varphi(x)$, and let $F_n\to E$; then the limit function $v_{E,\varphi}(z)=\lim_{n\to\infty} v_n(z)$, which exists for $z\varepsilon D$ and is independent of the choice of F_n, is called the generalized solution of the Dirichlet problem in region D relative to the set E and the function $\varphi(x)$. This function $v_{E,\varphi}(z)$ appears in the following result: If $\{f_n(z)\}$ is a sequence of polynomials, f_n of degree n, and if $\limsup_{n\to\infty}|f_n(x)|^{1/n}\leqq F(x)$ for each x of some set E, where $F(x)$ is a uniformly continuous function on E, then in the region D complementary to set E and containing the point $z=\infty$, $\limsup|f_n(z)|^{1/n}\leqq\exp\{g_E(z,\infty)+v_{E,\varphi}(z)\}$. Here $g_E(z,\infty)$ is a Green's function defined elsewhere [P. P. Korovkin, same Doklady (N.S.) **61**, 781–784 (1948); these Rev. **10**, 297]. A number of consequences are drawn from this result.
I. M. Sheffer (State College, Pa.).

Dinghas, Alexandre. Sur une inégalité concernant la croissance des fonctions harmoniques à plusieurs variables. C. R. Acad. Sci. Paris **233**, 126–127 (1951).

The author gives an improvement, based on Steiner symmetrization, of his previous estimate generalizing the Phragmèn-Lindelöf theorem to harmonic functions in several variables [same C. R. **232**, 1394–1395 (1951); these Rev. **12**, 825].
P. R. Garabedian (Stanford University, Calif.).

Górski, J. Sur certaines fonctions harmoniques jouissant des propriétés extrémales par rapport à un ensemble. Ann. Soc. Polon. Math. **23**, 259–271 (1950).

Leja avait étudié [Ann. Soc. Polon. Math. **12** (1933), 57–71 (1934)] de façon analogue au diamètre transfini des expressions qui, par une triple opération de max, min, lim, donnaient e^G (G fonction de Green d'un domaine). L'auteur part d'expressions analogues plus compliquées (avec deux fonctions arbitraires) et étudie des fonctions limites.
M. Brelot (Grenoble).

Schubert, Hans. Über ein gemischtes räumliches Randwertproblem der Potentialtheorie. I. Math. Nachr. **5**, 93–110 (1951).

We consider the uniform flow of an incompressible ideal fluid past a small obstacle in an infinite cylindrical channel, a section of whose surface is permitted to be free (pressure = const.), but is assumed undeformed. Linearization of Euler's equations leads to the Laplace equation for the acceleration potential $\psi=(p_\infty-p)/\rho$, (p_∞ = pressure p at infinity, ρ = constant pressure), in terms of which the author formulates the flow problem. The author's preference for this formulation over one by the more usual velocity potential lies in the fact (noted by Prandtl) that ψ can be taken a regular potential function in the entire flow region despite the presence of vortex sheets. He considers, as a model of this flow problem, the mixed boundary value problem for the potential ψ due to the presence of a pressure dipole on the axis (the flow induced by a horseshoe vortex of zero span); in this problem ψ is prescribed on the free surface and its normal derivative on the fixed walls of the cylinder. Closed solutions in terms of Bessel and Hankel functions are derived for the case the channel wall is entirely free or entirely fixed, the resulting formulas for downwash agreeing with those of Lotz [Luftfahrtforschung **12**, 250–264 (1935); **18**, 31 (1941)]. For the more difficult case of part free and part fixed boundary, the author reduces the problem to a

singular linear integral equation of the first kind the numerical solution of which appears in the literature.

D. Gilbarg (Bloomington, Ind.).

Birindelli, Carlo. Nuova trattazione di problemi al contorno di uno strato, per l'equazione di Poisson in tre variabili. Rivista Mat. Univ. Parma 2, 77–102 (1951).

This paper is the first part of a three part work on the solution of Poisson's equation, $\Delta U=f(x,y,z)$, in the slab $0<z<a$. On the two surfaces of the slab conditions of the form $\alpha_i U+\beta_i U_z=\gamma_i(x,y)$, $i=1,2$, are imposed. The α_i and β_i are non-zero constants, the γ_i are continuous, and f satisfies a Hölder condition. The method is a very close analogy to the author's treatment of the same problem in a two-dimensional strip [Ann. Mat. Pura Appl. (4) 25, 155–195 (1946); these Rev. 9, 355]. The extra dimension is taken care of by introducing cylindrical coordinates, ρ, θ, z, and making a Fourier analysis on θ. A solution is sought in the form $\sum w_j(z)g_k(\theta)U_{jk}(\rho)$, where the w_j are the eigenfunctions of the system $w''+\lambda w=0$, $\alpha_i w+\beta_i w'=0$ at 0 and a respectively. The g_k are the functions $\sin n\theta$ and $\cos n\theta$, and the U_{jk} are functions satisfying certain ordinary differential equations. In this first part, conditions on the behavior of U as $\rho\to\infty$ are determined under which the solution, if it exists, will be unique. Also sufficient conditions on the γ_i and f are found which ensure the convergence of the above series. The later parts of the paper are to deal with the question as to whether this series actually represents a solution to the problem. *J. W. Green.*

Flügge, S. Bemerkungen zum Potential eines homogen geladenen Rotationsellipsoids. Z. Physik 130, 159–163 (1951).

The conductor problem for an ellipsoid of revolution and the problem of determining the potential of a homogeneously charged solid ellipsoid of revolution (both classical problems) are solved as exercises in the use of ellipsoidal coordinates and the separation of variables. The treatment of the first problem is not essentially different from that in O. D. Kellogg, Foundations of Potential Theory [Springer, Berlin, 1929, p. 188]. The solution which is presented of the second problem is obtained by standard methods without the use of ingenious devices, which is in its favor. *J. W. Green.*

Booth, F. The solution of some potential problems in the theory of electrolytes. J. Chem. Phys. 19, 821–826 (1951).

An iterative method of solving the Poisson-Boltzmann equation for the potential in an electrolyte is presented. The method is to expand the potential in powers of a parameter involving the charge or charges on the boundary surfaces; the first term is the Debye-Hückel approximation and succeeding terms are obtained from the first by a recurrence formula. The radius of convergence of the series cannot be found. Examples with charged plane, cylindrical, and spherical boundary surfaces are given and are also of interest in the theory of colloids or emulsions. *T. E. Hull.*

Differential Equations

***Golubev, V. V. Lekcii po analitičeskoĭ teorii differencial'nyh uravneniĭ. [Lectures on the Analytic Theory of Differential Equations].** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 436 pp.

The purpose of the book is to present an exposition of mainly those aspects, now classic, of the analytic theory of differential equations (linear and nonlinear), which connect with the theory of algebraic functions, Riemann surfaces, the notion of genus of an algebraic function, multiple-valued functions, classification of singular points, and the notions of polyhedral and automorphic functions. In chapter I are presented the method of majorant functions, analytic continuation of solutions, and types of singular points (algebraic, transcendental, fixed, movable). Chapter II includes the Fuchsian singular point, Painlevé's theorem on integrals of $P(w',w,z)=0$, where P is a polynomial in w,w' and is analytic in z, Riemann surfaces, genus and its role, Hermite's theorem on the equation $P(w,w')=0$ without movable singular points, the inversions of the Schwarz-Christoffel formula, and the hyperelliptic equation. Chapter III deals with equations of order two with fixed singular points and the method of small parameters (Poincaré, Painlevé, Gambier). Linear equations are considered in chapter IV; this involves the regular singular point, the Riemann equation, groups of an equation, and the monodromic group. Chapter V covers in some detail the hypergeometric function and the problem of Riemann. In chapter VI are studied the differential equations satisfied by functions mapping polygonal regions on a circular region (the sides of the polygon being linear segments or circular arcs), the function of Schwarz, polyhedral functions, the modular function and its group. Chapter VII presents the theory of automorphic functions, mainly on the basis of the works in this field of H. Poincaré and F. Klein. Finally, in chapter VIII are given some applications of the preceding chapter: the geometry of Lobachevsky, the notion of the Fuchsian functions, uniformization of algebraic functions, functions and groups of Klein. The book is intended for advanced students in universities; it is clearly written and the material is well chosen.

W. T. Trjitzinsky (Paris).

Mikusiński, J. G.-. Un théorème d'unicité pour quelques systèmes d'équations différentielles considérées dans les espaces abstraits. Studia Math. 12, 80–83 (1951).

This paper is a continuation of earlier work by the same author [Ann. Soc. Polon. Math. 22, 157–160 (1949); these Rev. 12, 8]. The differential equations in question can be given the form $aX'(\lambda)=BX(\lambda)+F(\lambda)$, $a\neq 0$, where λ ranges over a real interval (α,β), X and F are functions with values in an n-dimensional vector space over a commutative ring A without divisors of zero, and B is an $n\times n$ matrix operator with entries from the ring. Differentiation of functions with values in the ring is any formally defined process postulated to satisfy certain of the usual rules. The uniqueness theorem is proved by induction, the case $n=1$ having been proved in the earlier paper mentioned above. *A. E. Taylor.*

Saltykow, N. Recherches sur l'ordre d'un système d'équations différentielles ordinaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 213–226 (1951).

The paper contains a discussion of the notion and determination of the "order" of a system of differential equations $f_i(x;y_1,y_1',\cdots,y_1^{(m_{1i})};\cdots;y_n,y_n',\cdots,y_n^{(m_{ni})})=0$, for $i=1,2,\cdots,n$. *P. Hartman* (Baltimore, Md.).

Bruwier, L. Sur l'équation récurrodifférentielle du premier ordre, de forme normale. Bull. Soc. Roy. Sci. Liège 20, 158–166 (1951).

This paper gives another proof of the existence of a sequence y_n satisfying the system $y_n'=F_n(x,y_n,y_{n+1})$,

$y_n(0)=y_0$, where the F's are continuous, have a common bound and satisfy a common Lipschitz condition.
J. M. Thomas (Durham, N. C.).

Bertolini, Fernando. Sulle soluzioni di un sistema di equazioni differenziali ordinarie. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 354–366 (1950) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 294 (1951).

The author gives an extremely complicated generalization of a theorem of Picone [Ann. Scuola Norm. Super. Pisa (2) 10, 13–36 (1941); these Rev. 3, 40] relating to bounds for the solutions of systems of ordinary differential equations. In this generalization it is not assumed that the systems are in the normal form. The principal theorem is used to obtain improvements of certain results due to the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 285–292 (1950); these Rev. 12, 179] relating to the length of an interval in which a solution of a linear differential equation has no zeros.
L. A. MacColl.

Leontovič, E. On the generation of limit cycles from separatrices. Doklady Akad. Nauk SSSR (N.S.) 78, 641–644 (1951). (Russian)

Let the system (1) $\dot{x}=P(x, y)$, $\dot{y}=Q(x, y)$ be of class $N>0$ in a certain plane region G. In G let the Jacobian $\Delta=D(P, Q)/D(x, y)$ be $\neq 0$ at all the critical points of (1). Consider in G the modified system (2) $\dot{x}=P+p$, $\dot{y}=Q+q$ where p, q and their first partials are arbitrarily small. It is known that upon passing from (1) to (2) new limit-cycles may arise: (a) from a complicated focus; (b) from a closed trajectory with zero characteristic number; (c) from a closed curve C which is a polygon whose sides are separatrices and whose vertices are saddle points. The first two cases have been considered more than once [see Andronov and Pontrjagin, C. R. (Doklady) Acad. Sci. URSS (N.S.) 14, 247–250 (1937); Andronov and Leontovič, ibid. (N.S.) 21, 423–426 (1938); Učenye Zapiski Gor'kovskogo Gosudarstv. Univ. 6, 3–24 (1939)]. The author discusses here (theorems stated without proofs) the case of a single separatrix S from a saddle point M back to S and obtains sufficient conditions for the generation of a preassigned number of limit-cycles in the neighborhood of $\bar{S}$. [Further relevant reference as regards the analysis: H. Dulac, Bull. Soc. Math. France 51, 45–188 (1923).]
S. Lefschetz (Princeton, N. J.).

Ważewski, T. Sur les systèmes de deux équations différentielles linéaires dont les intégrales tendent asymptotiquement vers une ellipse. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 41 (1948), 9–12 (1950). (French. Polish summary)

A theorem is stated concerning the asymptotic behavior of solutions of $y_i'=a_{i1}(t)y_1+a_{i2}(t)y_2+b_i(t)$, $i=1, 2$, as $t\rightarrow\infty$. The principal conclusions follow from results of Levinson [Duke Math. J. 15, 111–126 (1948); these Rev. 9, 509] for a system of n homogeneous equations.
F. M. Stewart.

Zlamal, Miloš. Oscillation criterions. Časopis Pěst. Mat. Fys. 75, 213–218 (1950). (English. Czech summary)

W. B. Fite [Trans. Amer. Math. Soc. 19, 344–352 (1918), p. 347] has considered the differential equation $y^{(n)}+q(x)y=0$, where n is a positive even integer and $q(x)>0$ and continuous for $x\geqq x_1$. He showed that if $\int^\infty q(x)dx=\infty$ all solutions of the differential equation are oscillatory neighboring $x=\infty$. This result appears not to have been known to A. Wintner who proved [Quart. Appl. Math. 7, 115–117 (1949); these Rev. 10, 456] a similar theorem for the case $n=2$ without however assuming that $q(x)$ is eventually positive. It was also not known to the reviewer who proved [Proc. Nat. Acad. Sci. U. S. A. 35, 192–193 (1949); see also Duke Math. J. 17, 57–61 (1950); these Rev. 11, 33, 248] that if $r(x)$ and $p(x)$ are continuous for x large and $r(x)>0$, the divergence of the integrals $\int^\infty dx/r(x)$ and $\int^\infty p(x)dx$ insures that all solutions of the self-adjoint differential equation (1) $[r(x)y']'+p(x)y=0$ are oscillatory near $x=\infty$.

The principal result of the present paper provides the following generalization of Fite's theorem. Suppose that $r(x)$ and $p(x)$ satisfy the above hypotheses. Then if $\int^\infty dx/r(x)=\infty$, and if there exists a positive function $g(x)$ of class C^1 such that $\int^\infty r(x)g'^2(x)dx/g(x)<\infty$ and $\int^\infty g(x)p(x)dx=+\infty$, the solutions of (1) are oscillatory. The choice $g(x)\equiv 1$ yields the reviewer's result referred to above, while the choice $r(x)\equiv g(x)\equiv 1$ yields Wintner's result and that of Fite ($n=2$).
W. Leighton (St. Louis, Mo.).

Bellman, Richard. On the asymptotic behavior of solutions of $u''-(1+f(t))u=0$. Ann. Mat. Pura Appl. (4) 31, 83–91 (1950).

It is shown that, if $f(t)\rightarrow 0$ and $\int^\infty |f(t)|^n dt<\infty$, the equation $u''-(1+f(t))u=0$ has a pair of solutions with the asymptotic forms $u\sim\exp(\pm\phi(t))$ as $t\rightarrow\infty$. For $n=2$, $\phi(t)=t+\frac{1}{2}\int_0^t f(t_1)dt_1$, a result due to Hartman [Trans. Amer. Math. Soc. 63, 560–580 (1948); these Rev. 9, 589]; for $n=3$,

$$\phi(t)=t+\frac{1}{2}\int_0^t f(t_1)dt_1-\frac{1}{2}\int_0^t f(t_1)\left[\int_{t_1}^t f(t_2)e^{2t_1-2t_2}dt_2\right]dt_1.$$

Extensions to higher order equations are indicated. The method of proof consists in obtaining a Riccati equation by $u'/u=v$ and converting this to a non-linear integral equation for v.
G. E. H. Reuter (Manchester).

Antosiewicz, H. A. A note on asymptotic stability. Quart. Appl. Math. 9, 317–319 (1951).

The author proves that if, in the real linear system of differential equations $x'=A(t)x$, the matrix of coefficients satisfies $\int^T \max_{|y|=1}(y\cdot A(t)y)dt\rightarrow-\infty$ as $T\rightarrow\infty$, then every solution satisfies $x(t)\rightarrow 0$ as $t\rightarrow\infty$. This is a particular case of a theorem on non-linear systems due to Wintner [Amer. J. Math. 68, 553–559 (1946); these Rev. 8, 272], whose proof is simpler than that of the author.
P. Hartman.

Haag, Jules. Sur certains systèmes différentiels à solution périodique lentement variable. Bull. Sci. Math. (2) 75, 15–21 (1951).

In an earlier paper [same Bull. (2) 71, 205–219 (1947); these Rev. 10, 195], the author studied, for small λ, the existence and stability of periodic solutions of the system $x'=\lambda f(x, t)$, where $x=(x_1, \cdots, x_n)$ and the functions f are periodic in t and are subject to general conditions, allowing a finite number of discontinuities in t on a period, for a fixed x. The results and methods of this paper are used to study the stable solutions, for small λ, of a system $x'=\lambda f(x, t, t)$, where the right-hand side is no longer periodic in t. In this system, the function $f(x, t, \theta)$, for a fixed value of θ, is subject to the conditions of the first paper and to a suitable restriction on the difference $|f(x, t, \theta)-f(y, t, \theta)|$.
P. Hartman (Baltimore, Md.).

McLachlan, N. W. Application of Mathieu's equation to stability of non-linear oscillator. Math. Gaz. 35, 105–107 (1951).

This is a routine stability investigation of Rayleigh's equation by the perturbation method [see chapter 6, J. J. Stoker, Nonlinear vibrations in mechanical and electrical systems, Interscience Publishers Inc., New York, 1950; these Rev. 11, 666]. *E. Pinney* (Berkeley, Calif.).

Litwiniszyn, J. A certain boundary problem of a vibrating string. Arch. Méc. Appl., Gdańsk 2, 75–88 (1950). (English. Polish summary)

The paper deals with the vibration of a homogeneous string of which the length changes with the time. The boundary conditions thus apply at one fixed point and at one variable point. The case of vibrating hoisting ropes is cited. Riemann's method is applied to make the integration.
R. E. Langer (Madison, Wis.).

Friedrichs, K. O. Criteria for discrete spectra. Comm. Pure Appl. Math. 3, 439–449 (1950).

The paper deals with the spectrum of characteristic values of an ordinary linear self-adjoint differential equation

$$\frac{d}{dx}\left\{p(x)\frac{d\varphi}{dx}\right\}-q(x)\varphi+\lambda r(x)\varphi=0,$$

with $p(x)>0$ and $r(x)>0$ in the interior of the basic interval of x. This interval may be finite or extend to infinity in either or both directions. In the former case it is supposed that $p(x)$, $q(x)$, $r(x)$, $1/p(x)$ and $1/r(x)$ do not all approach finite limits at the interval's end points. The irregular cases are thus in question. It is supposed that a spectrum is determined, boundary conditions or none being imposed as may be necessary to assure this. The exposition is in effect a summary of certain criteria upon $p(x)$, $q(x)$ and $r(x)$ which insure discreteness of the spectrum below a specific value of λ, or total discreteness of the spectrum. These are expressed mainly in terms of a function $h(x)$ which either vanishes or becomes infinite at each end of the interval and is such that $|dh/dx|=1/p(x)$ near these points, and in terms of the limiting behavior of the function

$$s(x)=\frac{1}{r(x)}\left\{q(x)+\frac{1}{4p(x)h^2(x)}\right\}.$$

The extension to partial differential equations is indicated.
R. E. Langer (Madison, Wis.).

Coddington, E. A., and Levinson, N. On the nature of the spectrum of singular second order linear differential equations. Canadian J. Math. 3, 335–338 (1951).

Let $p(x)\ (>0)$, $q(x)$ be real continuous functions on $0\leqq x<\infty$ such that $(py')'+(\lambda-q)y=0$ has at least one solution which is not of class $L_2(0,\infty)$. Let $S(\alpha)$ denote the spectrum of the differential operator associated with this differential equation and a homogeneous boundary condition $y(0)\cos\alpha+y'(0)\sin\alpha=0$ at $x=0$. Using formulae connected with Levinson's derivation [Duke Math. J. 18, 57–71 (1951); these Rev. 12, 828] of Titchmarsh's formula for the spectral resolution, the authors give proofs of (i) Weyl's result [Math. Ann. 68, 220–269 (1910), p. 251] that the set of cluster points S' of $S(\alpha)$ is independent of α and (ii) the result [Hartman and Wintner, Proc. Nat. Acad. Sci. U. S. A. 33, 376–379 (1947); Amer. J. Math. 71, 650–662 (1949); these Rev. 9, 435; 11, 109] that on any open interval on the complement of the closed set S', there exists a unique, regular, monotone increasing function $\alpha(\lambda)$ such that λ is in $S(\alpha(\lambda))$. It can be mentioned that assertions (i) and (ii), with the word "regular" replaced by "continuous", are simple corollaries of a result of the reviewer [Amer. J. Math. 71, 915–920 (1949); these Rev. 11, 438; for still another proof of (i) and (ii), given at the same time as the paper being reviewed, cf. Hartman and Wintner, Amer. J. Math. 72, 775–786 (1950); these Rev. 12, 717]. *P. Hartman.*

Iwata, Giiti. Non-hermitian operators and eigenfunction expansions. Progress Theoret. Physics 6, 216–226 (1951).

Eigenvalues and eigenfunctions for the Weber, hypergeometric, Legendre, Bessel, and confluent hypergeometric equations are given. Expansions of $1/(t-x)$ are considered and lead to eigenfunction expansions like Neumann's expansion in a series of Legendre or Bessel functions.
T. E. Hull (Vancouver, B. C.).

Couchet, Gérard. Sur l'équation complètement intégrale $P\,dx+Q\,dy+R\,dz=0$. Revue Sci. 89, 120–122 (1951).

Essentially the following theorem is stated: if (i) P, Q, R are polynomials in x, y, z, (ii) $(0,0,0)$ is a root of P, Q, R, (iii) $Pdx+Qdy+Rdz$ has an integrating factor, (iv) J is the Jacobian of P, Q, R, (v) J has rank 2 at $(0,0,0)$, and (vi) curl $(P,Q,R)\neq 0$ at $(0,0,0)$, then $P=Q=R=0$ imply $J=0$. The proof is based on geometrical considerations which do not seem even intuitively correct: the Jacobian's being of rank 2 at $(0,0,0)$ is said, for example, to imply that the three tangent planes to the surfaces $P=0$, $Q=0$, $R=0$ are distinct at $(0,0,0)$, whereas this is only necessarily true for two of three planes and the third may coincide with either of those or be indeterminate. *J. M. Thomas.*

Weinberg, Louis. Solutions of some partial differential equations (with tables). J. Franklin Inst. 252, 43–62 (1951).

Estrin, Thelma A., and Higgins, Thomas James. The solution of boundary value problems by multiple Laplace transformations. J. Franklin Inst. 252, 153–167 (1951).

Iterated Laplace transformations of functions of two or three variables are used in a purely manipulative manner to solve two boundary value problems. *R. V. Churchill.*

Sbrana, Francesco. Integrazione delle equazioni lineari alle derivate parziali del 2° ordine a coefficienti costanti in due variabili indipendenti col metodo degli operatori multipli. Atti Accad. Ligure 6, 224–239 (1950).

In earlier papers [same Atti 5, 7–33, 173–186, 187–200, 201–217 (1949); Boll. Un. Mat. Ital. (3) 4, 34–40 (1949); these Rev. 10, 701, 702] the author has developed a two-dimensional operational calculus based on the kernel $\exp\{s(\omega_1^2+\omega_2^2)^n\}$ with $n=1$. In the present paper he uses this operational calculus, both with $n=\frac{1}{2}$ and $n=1$, to derive solutions of various partial differential equations of the second order with constant coefficients. He makes it clear that the results are not altogether new and that the interesting feature of the paper is the method employed to deduce them. *A. Erdélyi* (Pasadena, Calif.).

Sbrana, Francesco, e Fumi, Fausto. Integrazione dell'equazione dei telegrafisti per mezzo degli operatori funzionali in una variabile. Atti Accad. Ligure 6, 273–298 (1950).

This is a counterpart of an earlier paper by the same authors [same Atti 5, 7–33 (1949); these Rev. 10, 701]. The equation is now $u_{xx}=au_{tt}+bu_t+cu$ (a, b, c constants).
A. Erdélyi (Pasadena, Calif.).

Karp, S. N. Wiener-Hopf techniques and mixed boundary value problems. Comm. Pure Appl. Math. **3**, 411–426 (1950).

The author summarizes his work as follows: In the present note the parallelism between the method of separation of variables and the Green's function integral equation method is shown to persist in the present situation as it does in problems of more classical type. This relationship leads to a characterization (from the standpoint of coordinate systems) of those problems in which the "Wiener-Hopf" type of problems (in an extended sense) arise. Certain heuristic advantages of the separation of variables procedure are also pointed out. *A. E. Heins* (Pittsburgh, Pa.).

Smirnov, M. M. Some nonhomogeneous boundary problems of the equation of heat conduction. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 367–370 (1951). (Russian)

The author constructs several solutions for the heat equation $\partial^2 u/\partial x^2 = \partial u/\partial t$ with the nonhomogeneous boundary conditions (1) $u_{x=0}=0$, $(\partial u/\partial x)_{x=l}=P=\text{constant}$ $(t=0)$, $u_{t=0}=0$ $(0<x<l)$; or (2) $u_{x=0}=Q=\text{constant}$, $(\partial u/\partial x)_{x=l}=0$ $(t>0)$, $u_{t=0}=0$ $(0<x<l)$. *C. G. Maple.*

Rubinšteĭn, L. I. On the propagation of heat in a stratified medium with varying phase state. Doklady Akad. Nauk SSSR (N.S.) **79**, 221–224 (1951). (Russian)

The linear heat conduction problems for a stratified medium with a fixed number of phase states is put into a more general form than has occurred in earlier mathematical literature. The author states that the method used by him for the solution of Stefan's problem [Doklady Akad. Nauk SSSR (N.S.) **58**, 217–220 (1947); these Rev. **9**, 287] can be extended to the more general problem stated here.

H. P. Thielman (Ames, Iowa).

Čarnyĭ, I. A. On methods of linearization of nonlinear equations of the type of the heat conduction equation. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1951**, 829–838 (1951). (Russian)

A method of linearization is given for certain non-linear differential equations arising in problems of heat conduction, of the filtration of gases through porous media, and of the non-stationary flow of ground waters. This method consists in approximating a non-linear coefficient in the differential equation by means of a function which is chosen in such a way that the equation is transformed into a linear one. For the purpose of illustrating this method let the problem of gas filtration be put in the form

$$\rho\mathbf{V} = c\rho\frac{d\rho}{dp}\operatorname{grad} p, \quad \operatorname{div}\rho\mathbf{V} = -m\,\partial\rho/\partial t,$$

where $\mathbf{V}$ is the velocity vector of filtration, c and m are constants depending on the nature of the gas, and the medium through which the filtration takes place, ρ is the variable density of the gas, p the absolute pressure, and t is time. In this case the linearization is accomplished by setting $\rho d\rho/dp$ equal to a constant, say α. This is equivalent to assuming that the density of the gas is given by an exponential expression in terms of the pressure p.

H. P. Thielman (Ames, Iowa).

Mann, W. Robert, and Wolf, František. Heat transfer between solids and gases under nonlinear boundary conditions. Quart. Appl. Math. **9**, 163–184 (1951).

The boundary condition, imposed by Newton's law of cooling on the problem of heat transfer between solids and gases, ceases to be linear if the film transfer factor changes with temperature. The authors state that it is their purpose to investigate this nonlinear boundary value problem under the most general physically significant relationship between the film transfer factor and the temperature. The problem is formulated for a semi-infinite solid. Since the temperature distribution $U(x, t)$ within the solid is completely determined by the surface temperature $U(0, t)$, the authors are able to reformulate the problem in terms of the function $U(0, t)$ of a single variable. The reformulation is accomplished by means of the Laplace transform. The problem is thus reduced to the solution of a nonlinear integral equation of the Volterra type in $U(0, t)$. Among the results obtained and explicitly stated are the following: For any film transfer factor of physical significance the heat transfer problem here considered has at least one solution $U(x, t)$ for all $x \geqq t$, $t>0$; if the film transfer factor satisfies a Lipschitz condition then $U(x, t)$ is unique. It is pointed out that the treatment used here for the solution of the nonlinear integral equation of the problem is applicable to more general nonlinear integral equations of the Volterra type.

H. P. Thielman (Ames, Iowa).

De Donder, Th. Simplification de la méthode d'intégration de Jacques Hadamard. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 960–961 (1950).

The author adds a remark to a previous paper [same vol., 545–547 (1950); these Rev. **12**, 708] pointing out that the finite part of one of the m-fold integrals (m odd) in his modified form of the Hadamard formulas can be written as the finite part of an $(m-1)$-fold integral. *D. C. Lewis.*

Višik, M. I. On strongly elliptic systems of differential equations. Doklady Akad. Nauk SSSR (N.S.) **74**, 881–884 (1950). (Russian)

Consider the system (of order $2m$) consisting of N linear partial differential equations in N unknown functions $u_1, \cdots, u_N$ of the n independent real variables $x_1, \cdots, x_n$:

$$\sum_{j=1}^{N}\sum_{(k)} a_{ij}^{(k_1, \cdots, k_{2m})}(x_1, \cdots, x_n)\frac{\partial^{2m}u_j(x_1, \cdots, x_n)}{\partial x_{k_1}\cdots\partial x_{k_{2m}}}+\cdots\cdot = f_i(x_1, \cdots, x_n), \quad i=1, \cdots, N,$$

where m, N, and n are positive integers; the summation is taken over all $2m$-tuples of positive integers, $(k_1, \cdots, k_{2m})$, with $1\leqq k_j\leqq n$ for $j=1, \cdots, 2m$; and $\cdots\cdot$ denotes a linear differential operator of $u_1, \cdots, u_N$ of order $\leqq 2m-1$. In matrix notation the system may be written

$$(*) \qquad Lu=\sum_{(k)} A^{(k_1, \cdots, k_{2m})}(x)\frac{\partial^{2m}u(x)}{\partial x_{k_1}\cdots\partial x_{k_{2m}}}+\cdots = f(x),$$

where $x=(x_1, \cdots, x_n)$, $A^{(k_1, \cdots, k_{2m})}(x)$ is the N by N matrix $\|a_{ij}^{(k_1, \cdots, k_{2m})}(x)\|$, and $f(x)=(f_1(x), \cdots, f_N(x))$. The system $(*)$ is said to be elliptic at a point x in the sense of I. G. Petrowsky [Mat. Sbornik N.S. **5**(47), 3–70 (1939); Uspehi Matem. Nauk (N.S.) **1**, no. 3–4, 44–70 (1946); these Rev. **1**, 236; **10**, 301] provided that the determinant

$$\Big|\sum_{(k)} A^{(k_1, \cdots, k_{2m})}(x)\xi_{k_1}\cdots\xi_{k_{2m}}\Big|$$

is not zero for any real numbers $\xi_1, \cdots, \xi_n$ such that $\xi_1^2+\cdots+\xi_n^2>0$. The author introduces the definition of a "strongly elliptic" system $(*)$, with the purpose of proving theorems for systems of equations which are analogous to

known theorems for the case of a single elliptic equation. A system (*) is called strongly elliptic at a point x provided that the N by N matrix

$$\sum_{(k)}[A^{(k_1,\cdots,k_{2m})}(x)+A^{*(k_1,\cdots,k_{2m})}(x)]\xi_{k_1}\cdots\xi_{k_{2m}}$$

is positive definite whenever the real numbers $\xi_1,\cdots,\xi_n$ satisfy $\xi_1^2+\cdots+\xi_n^2>0$, where $A^*(x)$ denotes the matrix obtained from $A(x)$ by interchanging rows and columns. A system (*) is said to be strongly elliptic on a set G if it is strongly elliptic at each point x of G. Clearly every strongly elliptic system is also an elliptic (Petrowsky) system, and the two definitions coincide in the case of a single equation, when $N=1$. Various theorems concerning strongly elliptic systems are announced: (1) the Dirichlet boundary value problem with zero boundary values (i.e.

$$u_i(x)\Big|_\Gamma=\frac{\partial u_i}{\partial n}(x)\Big|_\Gamma=\cdots=\frac{\partial^{m-1}u_i}{\partial n^{m-1}}\Big|_\Gamma=0,\quad i=1,\cdots,N,$$

where Γ is the boundary of a bounded open set G and $\partial/\partial n$ is the derivative in the normal direction) for the strongly elliptic system $Lu=f$ and the corresponding boundary value problem for the "adjoint" system $L^*v=g$ form a "Fredholm pair", in the sense that three theorems, analogous to Fredholm's three classical theorems about linear integral equations of the second kind, hold for this pair of systems; (2) for sufficiently small domains the zero boundary-value Dirichlet problem for the strongly elliptic system $Lu=f$ always has a (unique) solution, for any f; (3) the strongly elliptic operator L, considered over the class of all functions satisfying the zero boundary conditions, is semi-bounded; (4) the same operator L, considered over the same class of functions, has a discrete spectrum, and the operator $L-\lambda E$ (where E is the identity operator and λ is not an eigenvalue of L) has a completely continuous inverse. An example given by A. V. Bicadze [Uspehi Matem. Nauk (N.S.) 3, no. 6, 211–212 (1948); these Rev. 10, 300] shows that (2) and (3) do not hold for elliptic (Petrowsky) systems, while it is not known whether (1) and (4) hold for these systems. *J. B. Dias.*

van der Pol, Balth. On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 261–271, 272–284 (1951).

The differential equation in question is

$$\partial^2u/\partial s^2=2\partial u/\partial t-(\partial u/\partial s)^2$$

which is satisfied by $u=\log\theta_1(s/2\pi, it/2\pi)$. Expanding the partial derivatives involved, the author is naturally led to consider the functions

$$a_{2k-1}(t)=1+[2/\zeta(1-2k)]\sum_{l=1}^{\infty}e^{-lt}\sigma_{2k-1}(l)$$

which appear in the coefficients. Also considered are the functions $M_k=\theta_0^{4k}+\theta_2^{4k}+(-\theta_3^4)^k$ where $\theta_i=\theta_i(0, it/2\pi)$. Identities are obtained involving the $a_{2k-1}(t)$ and the M_k. These identities lead to numerous formulas, many already known, of various kinds. For example, there are formulas involving the $\sigma_{2k-1}(n)$ alone, formulas expressing $\tau(n)$ in terms of the $\sigma_{2k-1}(n)$, differential equations involving $J(\tau)$, and others. The author rather simply obtains the formula for $r_8(n)$ by first determining M_2 and then finding θ_3^8. Whether this procedure would simplify the computation of $r_{4k}(n)$ for $k>2$ is undecided. *H. S. Zuckerman.*

Integral Equations

Ganin, M. P. The equivalent regularizing operator for a system of singular integral equations. Doklady Akad. Nauk. SSSR (N.S.) 79, 385–387 (1951). (Russian)

The author considers the system

$$\text{(1)}\quad K\varphi(t)\equiv A(t)\varphi(t)+\frac{B(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau+\int_L K(t,\tau)\varphi(\tau)d\tau=g(t);$$

here L is a finite collection of simple, smooth, closed, non-intersecting curves; matrices A, B, K are assigned on L; g is a given vector on L; $\varphi(t)$ is the unknown vector. The functions are subject to Hölder conditions; it is assumed that the determinants $|A\pm B|$ are nonzero on L. An operator is an e.r. (equivalently regularizing) operator for (1) if it transforms (1) into an equivalent (regular) Fredholm system of integral equations. It is known that, when (1) is soluble, an e.r. operator can be constructed. *W. J. Trjitzinsky* (Paris).

Kveselava, D. A. Hilbert's boundary problem and singular integral equations in the case of intersecting contours. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 17, 1–27 (1949). (Russian. Georgian summary)

The author considers the singular integral equation

$$\text{(T)}\quad a(t)\varphi(t)+\frac{1}{\pi i}\int_L\frac{K(t,\tau)\varphi(\tau)}{\tau-t}d\tau=f(t),$$

when L consists of a finite number of closed and open piecewise smooth arcs having a finite number of common points; the hypotheses on the functions are in the nature of Hölder conditions. As noted by the author, the first study of this problem was made by the reviewer [Trans. Amer. Math. Soc. 60, 167–214 (1946); these Rev. 8, 211] who transferred to the case of arcs with common points some of the results of N. I. Muskhelišvili [same Trudy 10, 1–43, 161–162 (1941); these Rev. 4, 160]. The author develops further the theory of the problem (T) and of the related Hilbert problem (when the arcs of L have common points), on making use of a joint paper with Muskhelišvili [ibid. 11, 141–172 (1942); these Rev. 5, 269]; these developments involve the notion of index, classes of solutions, canonic solutions, and some theorems of Noether type. *W. J. Trjitzinsky.*

Mandžavidze, G. F. On a singular integral equation with discontinuous coefficients and its applications in the theory of elasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 279–296 (1951). (Russian)

The author investigates a singular integral equation

$$A(t_0)\varphi(t_0)+\frac{B(t_0)}{\pi i}\int_L\frac{\varphi(t)dt}{t-t_0}+\int_L k_1(t_0,t)\varphi(t)dt+\overline{\int_L k_2(t_0,t)\varphi(t)dt}=f(t_0),$$

containing in addition to the unknown function $\varphi(t)$ its complex conjugate. The functions $A(t)$, $B(t)$, $k_1(t_0,t)$, $k_2(t_0,t)$, $f(t_0)$ are permitted to have discontinuities of a certain type, and the contour L consists of a finite number of non-intersecting simple closed contours bounding a connected domain in the plane of the complex variable t. The results are applied to the solution of equations arising in

the fundamental mixed boundary value problem of plane elasticity, and are illustrated by the problem of deflection of a thin isotropic elastic plate (subjected to the action of normal load) part of whose boundary is clamped and part is free. *I. S. Sokolnikoff* (Los Angeles, Calif.).

***Chandrasekhar, S. Radiative Transfer.** Oxford University Press, 1950. xiv+393 pp. $7.00.

The book is concerned with the theory of radiative equilibrium in stellar and other atmospheres, and with a series of related mathematical problems.

In Chapter I the author discusses the properties of the radiation field in a medium that is capable of absorbing, emitting and scattering radiation, and derives the equation of transfer, which relates the local intensity of radiation to the "source function", i.e., the ratio of the coefficients of emission and absorption. In cases of perfect scattering, another equation can be obtained from the dependence of the source function on the scattering law, which expresses the angular dependence of the scattered radiation. From these two equations can be obtained an integro-differential equation, which is then transformed into an integral equation.

In Chapter II approximate solutions of this integral equation are discussed. The earliest and crudest solution is based on the Schuster-Schwarzschild idea of regarding the flow of radiation as consisting of two streams, one inward and one outward. Better approximations can be found by using a quadrature formula of Gauss type, a mathematical device that is equivalent, from the physical point of view, to the consideration of $2n$ streams of radiation flowing in $2n$ directions. The chapter also contains some discussion of quadrature formulae for the interval $(0, \infty)$ based on the zeros of the Laguerre polynomials, and formulae for functions possessing logarithmic singularities

In Chapter III an approximate solution is obtained by the above method for the case of isotropic scattering. Here, for the first time in the book, appears the approximate H-function, defined in the interval $(0, 1)$, and satisfying, as is proved in Chapter IV, an equation of the form

$$H(\mu)=1+\mu H(\mu)\sum_{j=1}^{n}\frac{a_j\Psi(\mu_j)}{\mu+\mu_j}H(\mu_j),$$

where the points μ_j are points of division for a Gauss quadrature formula, the numbers a_j are the corresponding coefficients, and $\Psi(\mu)$ is an even polynomial in μ such that $\int_0^1\Psi(\mu)d\mu\leqq\frac{1}{2}$.

In Chapter IV the first steps are taken towards exact solution of the integral equations. A non-linear equation for the scattering function $S(\mu, \varphi; \mu', \varphi')$ is obtained from the principle that the emergent radiation from an infinite plane-parallel atmosphere must be invariant under the addition of a further layer of arbitrary optical thickness. A linear integral equation connecting the intensity I with S is obtained, and it is found that S can be expressed in terms of a function $H(\mu)$ of a single variable μ, satisfying a non-linear integral equation of the form

$$\text{(1)}\qquad H(\mu)=1+\mu H(\mu)\int_0^1\frac{\Psi(\mu)}{\mu+\mu'}H(\mu')d\mu',$$

where $\Psi(\mu)$ depends on the form of the scattering law assumed.

Chapter V is devoted to the solution of this integral equation, including both Crum's solution in the form

$$\log H(z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log T(w)\frac{zdw}{w^2-z^2},$$

where

$$T(z)=1-2z^2\int_0^1\frac{\Psi(\mu)}{z^2-\mu^2}d\mu,$$

and numerical iteration methods of claculating $H(\mu)$ for given scattering laws.

Most of the remainder of the book is concerned with the application and generalization of these ideas, which form the mathematical key to the author's treatment of the subject. Among the topics discussed are: general scattering laws (chap. VI); atmospheres of finite optical thickness, in which case (1) is replaced by a pair of integral equations for two functions $X(\mu)$ and $Y(\mu)$ (chaps. VII–IX); Rayleigh scattering and scattering by planetary atmospheres, including a discussion of the effects of polarisation (chap. X); special problems in stellar atmospheres (chap. XI); the effects of scattering and absorption on the formation of spectral lines, the softening of radiation by multiple Compton scattering, and the broadening of lines by electron scattering (chap. XII); and a further mathematical discussion of the functions $H(\mu)$, exhibiting them as the Laplace transforms of solutions of linear integral equations of the Schwarzschild-Milne type (chap. XIII). *F. Smithies.*

Kourganoff, Vladimir. Exact source functions by an extension of Chandrasekhar's limiting process. Astrophys. J. 113, 419–431 (1951).

The reviewer has established [Radiative Transfer, Oxford, 1950, §§39, 43, 59; see the preceding review] a certain correspondence theorem which enables one to write the exact solutions (by "passing to the limit of infinite approximation") for the emergent intensities from the form of the solutions obtained in a finite approximation in which the integro-differential equations representing the equation of transfer is replaced by a system of linear equations with constant coeficients. By considering two problems in isotropic scattering in semi-infinite atmospheres the author shows how the exact solutions for the source functions can also be obtained by a similar correspondence theorem. The solutions obtained in this manner are in agreement with those which can be obtained in the cases considered by applying an inverse Laplace transformation to the known expressions giving the emergent intensities.
S. Chandrasekhar (Williams Bay, Wis.).

Drăganu, Mircea. La résolution de l'équation intégro-différentielle de la diffusion des neutrons à l'aide des équations intégrales. J. Math. Pures Appl. (9) 29, 141–168 (1950).

This paper deals with the problem of transfer in a plane-parallel atmosphere scattering isotropically with an albedo less than one. It is assumed that there are sources of radiation distributed through the medium and solutions are sought for given angular distributions of radiation incident on the two surfaces. An integral equation of the Schwarzschild-Milne type appropriate for the problem is derived and a method of solution is described which is essentially equivalent to that of analyzing the emergent radiation into first order, second order and higher order scatterings. [The author does not seem to be aware that a large class of problems of the type he considers has been exactly solved. Thus, in the case when there are no sources distributed in the medium, the complete solution can be expressed in terms of the standard problem of diffuse reflection and transmission for which the exact solution is known [cf. S. Chan-

drasekhar, Radiative Transfer, Oxford, 1950, Chapter IX; second preceding review].] *S. Chandrasekhar.*

Davison, B. Angular distribution of neutrons at the interface of two adjoining media. Canadian J. Research. Sect. A. 28, 303–314 (1950).

The equation of transfer

$$(1) \qquad \mu dI(\tau, \mu)/d\tau = I(\tau, \mu) - \tfrac{1}{2}\tilde{\omega} \int_{-1}^{+1} I(\tau, \mu')d\mu',$$

where $\tilde{\omega}$ is a constant less than 1, admits a solution of the form (2) $I(\tau, \mu) = L_0 e^{k\tau}/(1-k\mu)$, where L_0 is a constant and k is the positive root (<1) of the characteristic equation $\tilde{\omega} \log [(1+k)/(1-k)] = 2k$. It is known that the solution of (1) which has the asymptotic form (2) for $\tau \to \infty$ and satisfies the boundary condition $I(0, -\mu) = 0$ $(0 < \mu \leqq 1)$ at $\tau = 0$ has the form (3) $I(0, +\mu) = [L_0/H(1/k)][H(\mu)/(1-k\mu)]$ $(0 < \mu \leqq 1)$, where $H(\mu)$ is the H-function defined for the characteristic function $\frac{1}{2}\tilde{\omega}$ [cf. S. Chandrasekhar, Radiative Transfer, Oxford, 1950, Chap. V; third preceding review]. In this paper the author considers the modification of the solution (3) when the half-space above $\tau = 0$ is filled with a medium with an "albedo" $\tilde{\omega}_1$ different from $\tilde{\omega}$. By using integral equations of the Schwarzschild-Milne type appropriate to this problem, the author shows that the angular distributions of the outward, $I_+(\mu)$, and the inward, $I_-(\mu)$, directed radiations at $\tau = 0$ are given by

$$I_+(\mu) = L_0 \frac{H_1(1/k)}{H(1/k)} \frac{H(\mu)}{H_1(\mu)(1-k\mu)},$$

and

$$I_-(\mu) = L_0 \frac{H_1(1/k)}{H(1/k)} \frac{H_1(\mu)}{H(\mu)(1+k\mu)},$$

where $H_1(\mu)$ is the H-function defined for the characteristic function $\frac{1}{2}\tilde{\omega}_1$. It is also proved that

$$\frac{\int_0^1 [I_+(\mu) - I_-(\mu)]\mu d\mu}{\int_0^1 [I_+(\mu) + I_-(\mu)] d\mu} = k^{-1}(1-\tilde{\omega}_1)^{\frac{1}{2}}(1-\tilde{\omega}_2)^{\frac{1}{2}}.$$

[The results of this paper can be obtained by entirely elementary methods when one observes that $I_-(\mu)$ must result from the reflection of $I_+(\mu)$ by the semi-infinite atmosphere $(\tilde{\omega} = \tilde{\omega}_1)$ overlying $\tau = 0$ while the reflection of $I_-(\mu)$ by the atmosphere $(\tilde{\omega} = \tilde{\omega}_2)$ below $\tau = 0$ must give

$$I_+(\mu) - L_0 H(\mu)/\{H(1/k)(1-k\mu)\}.]$$

S. Chandrasekhar (Williams Bay, Wis.).

Chandrasekhar, S. The angular distribution of the radiation at the interface of two adjoining media. Canadian J. Physics 29, 14–20 (1951).

B. Davison [Canadian J. Research. Sect. A. 28, 303–314 (1950); see the preceding review] solved the following problem in the theory of radiative transfer: The half-space ($\tau > 0$) above a semiinfinite plane-parallel isotropically scattering atmosphere with an albedo $\tilde{\omega}_1$ is filled with a similar atmosphere with a different albedo $\tilde{\omega}_2$. The equations of transfer are

$$(1) \qquad \mu dI(\tau, \mu)/d\tau = I(\tau, \mu) - \tfrac{1}{2}\tilde{\omega}_1 \int_{-1}^{1} I(\tau, \mu')d\mu' \quad (\tau > 0);$$

$$(2) \qquad \mu dI(\tau, \mu)/d\tau = I(\tau, \mu) - \tfrac{1}{2}\tilde{\omega}_2 \int_{-1}^{1} I(\tau, \mu')d\mu' \quad (\tau < 0).$$

It is known that (1) admits an integral of the form (3) $I(\tau, \mu) = L_0 e^{k_1\tau}/(1-k_1\mu)$ where k_1 $(0 < k_1 < 1)$ satisfies (4) $\tilde{\omega}_1 \log [(1+k_1)/(1-k_1)] = 2k_1$. The problem is to find $I(0, \pm\mu)$ $(0 < \mu \leqq 1)$ for the solution of equations (1) and (2) which has the asymptotic behaviour (3) as $\tau \to +\infty$ and which vanishes as $\tau \to -\infty$. Davison solved this by using the integral equations of the Schwarzschild-Milne type which govern the problem. The author now shows that all of Davison's results can be obtained in an elementary way by using certain invariances of the problem developed by him in recent years and summarized in his book [Radiative Transfer, Oxford, 1950; see the fourth preceding review].

E. T. Copson (St. Andrews).

Halpern, Otto, and Luneburg, R. K. Multiple scattering of neutrons. II. Diffusion in a plate of finite thickness. Physical Rev. (2) 76, 1811–1819 (1949).

The problem of diffuse reflection and transmission of a parallel beam of neutrons by a plate of finite thickness is considered by obtaining a pair of integral equations for the reflected and the transmitted intensities. Asymptotic formulae for large thicknesses are also given. [The methods used are equivalent to those well known in the solution of the corresponding problem of diffuse reflection and transmission by a plane parallel atmosphere of finite optical thickness [S. Chandrasekhar, Radiative Transfer, Oxford, 1950, chapters VII, VIII and IX; fifth preceding review; Astrophys. J. 106, 152–216 (1947); 107, 48–72, 188–215 (1948); these Rev. 9, 444, 593]. Thus the pair of integral equations derived by the authors is equivalent to the X and Y equations and the solution given for the transmitted intensity is identical with that given in the references cited.] *S. Chandrasekhar* (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

Bourbaki, Nicolas. Sur certains espaces vectoriels topologiques. Ann. Inst. Fourier Grenoble 2 (1950), 5–16 (1951).

Many of the results of Dieudonné and Schwartz [same Ann. 1, 61–101 (1950); these Rev. 12, 417] can be generalized to one or the other of what the author defines as "tonnelé" and "bornologique" spaces. Let E be a locally convex topological space. A subset T of E is called a "tonneau" if T is convex, closed, circular, and if E is the linear extension of T. E is called a "tonnelé" space if all tonneaux are neighborhoods of 0. If E and F are locally convex topological spaces, $\mathfrak{L}(E, F)$ is defined as the set of all linear transformations on E to F. If in addition E is tonnelé, then it is shown that all subsets H of $\mathfrak{L}(E, F)$ which are bounded in the weak topology are equicontinuous. It is also shown that if A is a convex, circular, bounded, and complete subset of E, then A is "absorbed" by each tonneau T (i.e. for each T there exists an α such that $\alpha A \subset T$). A subset A of E is called "bornivore" if it is convex, circular, and absorbs all bounded subsets of E. A space E is said to be "bornologique" if all bornivore subsets are neighborhoods of 0. This type of space has been previously considered by Mackey [Trans. Amer. Math. Soc. 60, 519–537 (1946); these Rev. 8, 519]. The author shows that a necessary and sufficient condition that a space be bornologique is that its topology be $\tau(E, E')$ and that all linear functionals on E bounded on each bounded subset of E be continuous. From

these theorems the author obtains several interesting corollaries. The paper concludes with a discussion of bilinear functions. *R. Phillips* (Los Angeles, Calif.).

Schwartz, Laurent. Un lemme sur la dérivation des fonctions vectorielles d'une variable réelle. Ann. Inst. Fourier Grenoble **2** (1950), 17–18 (1951).

Let E be a real linear space and T_1, T_2 topologies for E such that T_1 is finer than T_2 and (E, T_i) is a linear topological space. Suppose also that (P) for each T_1-neighborhood V_1 of ϕ there is a T_1-neighborhood W_1 of ϕ such that V_1 contains the T_2-closed convex hull of every T_1-compact subset of W_1. Let f be a mapping of the reals into E whose T_2-derivative f' exists and is T_1-continuous. Under these hypotheses, the author proves that f' is also the T_1-derivative of f. He remarks that (P) holds if T_2 is the weak topology associated with T_1, and also if E is T_1-complete. *V. L. Klee, Jr.* (Princeton, N. J.).

Köthe, Gottfried. Die verschiedenen Reziproken einer unendlichen Matrix. Monatsh. Math. **55**, 153–156 (1951).

This paper and the review both use the concepts and notations of G. Köthe and O. Toeplitz [J. Reine Angew. Math. **171**, 193–226 (1934), cited as KT], G. Köthe [Math. Ann. **114**, 99–125 (1937), cited as K-1; J. Reine Angew. Math. **178**, 193–213 (1938), cited as K-2; see also Math. Nachr. **4**, 70–80 (1951); these Rev. **12**, 615]. The principal result is the following theorem: Let λ and μ be two perfect (vollkommen) spaces, $\Sigma(\lambda)$ and $\Sigma(\mu)$ the corresponding rings of matrices of continuous linear maps (in the weak sequential topology; see KT) of the appropriate space into itself. Let $\mathfrak{A}\epsilon\Sigma(\lambda)\cap\Sigma(\mu)$ and let $\mathfrak{B}$ be the twosided inverse of $\mathfrak{A}$ in $\Sigma(\lambda)$, $\mathfrak{C}$ the twosided inverse of $\mathfrak{A}$ in $\Sigma(\mu)$. Then $\mathfrak{B}=\mathfrak{C}$ if and only if $\mathfrak{A}(\lambda\cap\mu)$ is dense in $\lambda\cap\mu$ (in the weak topology; see K-2). If $\mathfrak{B}=\mathfrak{C}$, then $\mathfrak{A}\epsilon\Sigma(\lambda\cap\mu)$ (and also in $\Sigma((\lambda\cup\mu)^{**})$ and $\mathfrak{A}$ has the twosided inverse $\mathfrak{B}$ in these rings. An example is constructed exhibiting a matrix $\mathfrak{A}$ and twosided inverses $\mathfrak{B}$ and $\mathfrak{C}$ such that $\mathfrak{B}\neq\mathfrak{C}$ (and thus $\overline{\mathfrak{A}(\lambda\cap\mu)}\neq\lambda\cap\mu$). The straightforward proof of the theorem utilizes properties of the dual space λ^* of λ (see KT and K-1), as well as properties of the transpose $\mathfrak{A}'$ of $\mathfrak{A}$ (defined on λ^* if $\mathfrak{A}$ is defined on λ; see the above references). *G. K. Kalisch.*

Ruston, A. F. On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach space. Proc. London Math. Soc. (2) **53**, 109–124 (1951).

The equivalent of a Fredholm equation of the second kind for a general linear operator K is $x=y+\lambda Kx$ where y is a given element of a Banach space B. The operators K considered in this paper are in the "trace class" defined as follows. Let γ be the "greatest crossnorm" of Schatten and form $B\otimes_\gamma B^*$ [cf. Schatten, A theory of cross-spaces, Princeton Univ. Press, 1950; these Rev. **12**, 186]. The finite expressions $\sum x_i\otimes f_i$ correspond to transformations on B with a finite-dimensional range and this correspondence can be extended to associate with the elements of $B\otimes_\gamma B^*$ the operators K of the "trace class". For finite expressions $\mathrm{tr}(\sum x_i\otimes f_i)=\sum f_i(x_i)$ has the properties of a trace for the corresponding operator and the above construction can be used to introduce a trace, $\mathrm{tr}(K)$, for K. To solve the operator equation, the author defines a numerically valued function $d(\lambda)$ of λ and a function $D(\lambda)=\sum_n\lambda^nD_n$ where the D_n are transformations in the trace class and $D(\lambda)$ is a bounded operator for all values of λ. For these the identity $D(\lambda)=d(\lambda)+\lambda KD(\lambda)$ holds. Thus, when $d(\lambda)\neq 0$, $x=d(\lambda)^{-1}D(\lambda)y$ is a solution of the given operator equation. *F. J. Murray* (New York, N. Y.).

Bourgin, D. G. Classes of transformations and bordering transformations. Bull. Amer. Math. Soc. **57**, 223–237 (1951).

An exposition of published and unpublished work on stability problems for transformations from one Banach space to another is given. The first problem is whether, if T takes the space into itself so that $\|T(x+y)-Tx-Ty\|<\epsilon_1$, there exists an additive U so that $\|Tx-Ux\|<\epsilon_2$. For $\epsilon_1=\epsilon_2$ constant, the existence was proved by Hyers [Proc. Nat. Acad. Sci. U. S. A. **27**, 222–224 (1941); these Rev. **2**, 315]. Possible extensions with $\epsilon_1=\epsilon_1(\|x\|, \|y\|)$, $\epsilon_2=\epsilon_2(\|x\|)$ are discussed. The second problem is whether for some k, if T takes X into itself so that $|\|Tx-Ty\|-\|x-y\||<\epsilon$ then $\|Tx-Ux\|<k\epsilon$ for some isometry U. For Hilbert space and the space of functions continuous over a compactum this has been answered by Hyers and Ulam [Bull. Amer. Math. Soc. **51**, 288–292 (1945); Ann. of Math. (2) **48**, 285–289 (1947); these Rev. **7**, 123; **8**, 588] and by the author for a class of uniformly convex spaces including $L_p(0, 1)$ [Bull. Amer. Math. Soc. **52**, 704–714 (1946); these Rev. **8**, 157]. [Reviewer's note: The author states these problems as particular cases of a general problem for uniform spaces; the specializations are incorrect, the first trivially, the second essentially.] Conditions under which isometrical transformations on a metric group are automorphisms and related problems are discussed, followed by a result on approximately convex functions. A number of unpublished results on exactly and approximately multiplicative transformations on spaces of continuous functions, and their bearing on the homeomorphism of the argument spaces and Banach isomorphism of the function spaces follow. In the final sections transformations homotopic to some standard type are discussed, particularly in relation to the problems of defining an index for transformations on non-compact sets, and of fixed points and fixed point classes for such transformations. *J. L. B. Cooper* (Cardiff).

Phillips, R. S. A note on ergodic theory. Proc. Amer. Math. Soc. **2**, 662–669 (1951).

This note answers a number of questions raised by the reviewer in Functional Analysis and Semi-Groups [Amer. Math. Soc. Colloq. Publ. v. 31, New York, 1948; these Rev. **9**, 594; cited as H in the following]. Assume (i) that $T(\xi)$ is a strongly measurable semi-group of linear bounded operators on a complex B-space to itself, (ii$_s$) $\exp(-\lambda\xi)\|T(\xi)x\|\,\epsilon\, L(0, \infty)$, $\lambda>0$, for all x [or (ii$_u$) $\exp(-\lambda\xi)\|T(\xi)\|\,\epsilon\, L(0, \infty)$], (iii) $\eta^{-1}\int_0^\eta T(\xi)xd\xi\rightarrow x$ for all x as $\eta\rightarrow 0$. The author proves the conjecture [H, p. 301] that if (i), (ii$_u$), (iii) hold, if the range of A^2 is closed, and if $\lim_{\lambda\rightarrow 0}\lambda^2R(\lambda; A)x=0$ for all x, then $T(\xi)$ is Abel-ergodic at infinity in the uniform topology. Next he shows by an elementary argument, that if (i), (ii$_s$) hold, if $T(\xi)$ is strongly Abel-ergodic at 0, and if the Cesàro averages of order α are strongly bounded for $0<\xi<1$, then $T(\xi)$ is also strongly $(C-\alpha)$-ergodic at 0. If $\mathfrak{R}[A]$ is the range of A, $\mathfrak{Z}[A]$ the zero manifold, let $\mathfrak{X}_1$ be the closure of $\mathfrak{R}[A]\oplus\mathfrak{Z}[A]$, A_1 the retraction of A on $\mathfrak{X}_1$, and A_2 the retraction of A on the closure of $\mathfrak{R}[A]$. Now if $T(\xi)$ satisfies (i), (ii$_s$), (iii), and if $\|\lambda R(\lambda; A)\|\leqq M$ for $0<\lambda<1$, then $\lim_{\lambda\rightarrow 0}\lambda R(\lambda; A)x=x$ for all x in $\mathfrak{X}_1$, $T(\xi)\mathfrak{X}_1\subset\mathfrak{X}_1$, and $\mathfrak{R}[A]$, $\mathfrak{R}[A_1]$, and $\mathfrak{R}[A_2]$ have

identical closures. Further $\lambda=0$ is either in the resolvent set or in the continuous spectrum of A_2. If $\mathfrak{X}_1=\mathfrak{X}_2$, $T(\xi)$ is strongly Abel-ergodic at infinity. The author disproves a conjecture [H, p. 295] concerning the ergodic character of $T(\xi)$ when $\lambda=0$ is not in the residual spectrum of either A or A_2 by exhibiting a counter example. Another example shows that conditions (i), (ii_a) and (iii) are consistent with $\lim\sup_{\xi\to 0}\|T(\xi)\|=\infty$. Finally, an example is given of a strongly continuous group operator $T(\xi)$ such that $\|T(\xi)\|$ is discontinuous [H, p. 184]. *E. Hille.*

Williamson, J. H. Spectral representation of linear transformations in ω. Proc. Cambridge Philos. Soc. **47**, 461–472 (1951).

Let ω be the space of all sequences of complex numbers and φ be its dual space. In the usual topology for ω, the algebra A of continuous linear operators of ω into ω is isomorphic to the algebra of row-finite (i.e., only a finite number of non-zero terms in each row) matrices if customary conventions are employed. Call $T\varepsilon A$ adequately restricted (a.r.) if $\lim\sup_{n\to\infty}|u(T^N x)|^{1/n}<\infty$ for $x\varepsilon\omega$, $u\varepsilon\varphi$ and designate by $)T($ the sup of this expression over $x\varepsilon\omega$, $u\varepsilon\varphi$. Call T u.a.r. if $)T(<\infty$ and q.n. if $)T(=0$. The author shows that if T is a.r. then there is a non-empty countable set $\lambda_1, \lambda_2, \cdots$ of complex numbers and corresponding sets $P_1, P_2, \cdots; Q_1, Q_2, \cdots$ in A where these are u.a.r. and the Q_i are q.n. such that the P_i and Q_j commute with each other and with T and fulfill the relations $P_iP_j=\delta_{i,j}P_i$, $P_iQ_j=\delta_{i,j}Q_i$, $Q_iQ_j=\delta_{i,j}(T-\lambda_i)Q_i$, $I=\sum P_i$, $T=\sum(\lambda_iP_i+Q_i)$. Furthermore, the λ's, P's and Q's are unique with these properties, the λ's constituting the spectrum of T as an operator in A. The author recasts this representation into an integral form by setting $Q=\sum Q_i$ and $E(\lambda)=\sum_{\lambda_i\leq\lambda}P_i$. Then $T=\int dE(\lambda)+Q$ and $I=\int dE(\lambda)$ where Q is q.n. He shows that, under reasonable conditions, only T which are a.r. have a spectral representation of this type. Properties of the spectrum for T a.r. or otherwise are also discussed. *B. Yood* (Ithaca, N. Y.).

✶**Lorch, E. R. Normed rings—the first decade.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 249–258. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Expository paper.

Šilov, G. E. Homogeneous rings of functions. Uspehi Matem. Nauk (N.S.) **6**, no. 1(41), 91–137 (1951). (Russian)

The present paper presents another chapter in the already extensive theory of commutative Banach algebras. The paper is divided into six §§, whose contents may be summarized as follows. In §1, certain essential preliminaries are described. Let G be a compact Abelian group, written additively, and let L be a complex Banach space (which may be a commutative Banach algebra or the complex number field), with norm denoted by $|\cdot|$. Consider a complex linear space R of L-valued continuous functions on G, addition and scalar multiplication being defined pointwise. Suppose that R admits a norm $\|\cdot\|$ which need have no connection with the norm $|\cdot|$. Suppose further that for $f(t)\varepsilon R$ and $h\varepsilon G$, the translate $f(t+h)$ belongs to R, and that for all $h\varepsilon G$, there exists a constant $C_h\geqq 0$ such that $\|f(t+h)\|\leqq C_h\cdot\|f(t)\|$ for all $f\varepsilon R$. Such a space of L-valued functions is called a homogeneous space of functions. If convergence in the norm $\|\cdot\|$ of a sequence $\{f_n(t)\}_{n=1}^{\infty}$ implies the convergence in $|\cdot|$ of $\{f_n(t)\}_{n=1}^{\infty}$ for each fixed $t\varepsilon G$, then the operator $f(t)\to f(t+h)$ is necessarily bounded. A function $f(t)\varepsilon R$ is translation-continuous if for every $\epsilon>0$, there exists a neighborhood $U(0)$ in G such that $k\varepsilon U(0)$ implies $\|f(t+k)-f(t)\|<\epsilon$. The author first proves that every homogeneous space of functions R which contains a dense set of translation-continuous elements must (a) consist entirely of translation-continuous functions and (b) admit a norm $|||\cdot|||$ equivalent to $\|\cdot\|$ for which $|||f(t+h)|||=|||f(t)|||$ for all $h\varepsilon G$. For the case L = the complex numbers and R containing a dense set of continuous characters (which are obviously translation-continuous), assertions (a) and (b) apply.

In §2, a homogeneous space R of L-valued functions is considered which satisfies (a) and (b) of the preceding paragraph. The L-valued integral $\int f(t)dt$ (dt representing Haar measure on G) exists, in any of a number of senses, for all $f\varepsilon R$. Let $X=\{\chi_\alpha\}$ be the character group of G. The αth Fourier coefficient $C_\alpha(f)$ of $f\varepsilon R$ is defined as $\int f(t)\overline{\chi_\alpha(t)}dt$, and is an element of L. It is proved that $C_\alpha(f)\chi_\alpha(t)\varepsilon R$. A brief proof is then given for the theorem of Bochner and von Neumann generalizing Fejér's theorem on trigonometric series [Trans. Amer. Math. Soc. 37, 21–50 (1935)], which shows how $f(t)$ can be reconstructed from the elements $C_\alpha(f)$ and the characters $\chi_\alpha(t)$. A corollary is that $C_\alpha(f)=0$ for all α implies $f=0$. Also, the Riemann-Lebesgue lemma is generalized by showing that for every $\epsilon>0$, only a finite number of the elements $C_\alpha(f)$ have norms exceeding ϵ.

The contents of §3 are taken in toto from an earlier treatise by the author [Trudy Mat. Inst. Steklov 21 (1947); these Rev. 9, 596], to the review of which we refer for terminology not explained here.

In §4, the author introduces the following concepts. Let G, X be as above and let K be a primary commutative Banach algebra with the sole maximal ideal Q, and with norm $\|\cdot\|$. Let ω be a homomorphism of X into K such that $(\omega(\chi)-e)\varepsilon Q$ for all $\chi\varepsilon X$. Consider all complex linear combinations $f(t)=\sum_{j=1}^{n}C_j\chi_j(t)$ of characters of G, Let

$$|||f|||=\sup_{t\varepsilon G}\|C_1\chi_1(t)\omega(\chi_1)+\cdots+C_n\chi_n(t)\omega(\chi_n)\|.$$

Let functions f be added and multiplied pointwise. With the norm $|||\cdot|||$, one obtains a normed algebra, whose completion is called a continuous direct sum of the algebras K by the group G and is denoted by $K_\omega(G)$. It is easy to see that convergence of a sequence of functions of the form $\sum_{j=1}^{n}C_j\chi_j(t)$ in the norm $|||\cdot|||$ implies uniform convergence on G. Hence $K_\omega(G)$ is realized as an algebra of continuous functions on G. It is proved that the points t of G, which correspond in the canonical way with certain maximal ideals in $K_\omega(G)$, are in reality all of the maximal ideals in $K_\omega(G)$. The term "continuous sum of algebras K" is now justified by representing $K_\omega(G)$ as an algebra of K-valued functions on G. Thus, the complex-valued function $\sum_{j=1}^{n}C_j\chi_j(t)=f(t)$ may be replaced by the K-valued function $\sum_{j=1}^{n}C_j\chi_j(t)\omega(\chi_j)=\bar{f}(t)$. If a sequence $\{f_n(t)\}_{n=1}^{\infty}$ converges in the norm $|||\cdot|||$ to $X(t)$, then, by definition, the sequence $\{\bar{f}_n(t)\}_{n=1}^{\infty}$ must converge in the norm $\|\cdot\|$ in K for every fixed $t\varepsilon G$, and thus represents a K-valued function $x(t)$ on G. The norm $|||\bar{x}|||=\sup_{t\varepsilon G}\|\bar{X}(t)\|$ is equal to $|||x|||$ in $K_\omega(G)$. The Banach algebra obtained in this wise is denoted by $\bar{K}_\omega(G)$. It is isomorphic to $K_\omega(G)$. Applying the results of §1, the author shows that every element of $K_\omega(G)$ is translation-continuous, and by applying the results of §2, that $K_\omega(G)$ (and $\bar{K}_\omega(G)$) have nil radical. If $K_\omega(G)$ is regular, then it is of type C.

In §5, the author considers homogeneous algebras of functions, that is, algebras R of continuous complex functions on compact Abelian groups G which are homogeneous spaces as defined in the first paragraph and for which every $f \epsilon R$ not vanishing on G has an inverse in R. It is proved that every such algebra contains all continuous characters of G and that these are in fact a set of generators for G. It is next proved that every homogeneous algebra of functions is realizable as an algebra $K_u(G)$. The author also identifies all subrings of a homogeneous algebra of functions invariant under translations: each is generated by a subgroup of X. The paper closes with the discussion of a number of examples, among them "Wiener algebras", i.e., all functions $\sum_\alpha a_\alpha \chi_\alpha$ such that $\sum_\alpha |a_\alpha| < \infty$.

E. Hewitt (Upsala).

Koehler, Fulton. Note on a theorem of Gelfand and Šilov. Proc. Amer. Math. Soc. 2, 541–543 (1951).

The theorem discussed is the following. Let K be a complex commutative Banach algebra with a unit. Suppose that K fulfills the condition that for each $x \epsilon K$ there exists $y \epsilon K$ such that $x(M) = \overline{y(M)}$ for all $M \epsilon \mathfrak{M}$ where $\mathfrak{M}$ is the set of maximal ideals of K. If $\mathfrak{M}$ is given its usual topology, then the set of functions $x(M)$ is dense in the space of all continuous functions on $\mathfrak{M}$. The author notes that Gelfand and Šilov [Mat. Sbornik N.S. 9(51), 25–39 (1941); these Rev. 3, 52] use other topologies for $\mathfrak{M}$ in the course of the proof and then proceeds to give a proof wherein $\mathfrak{M}$ has only the original topology. It should be noted that this can be accomplished by simply referring to the complex case of the Stone generalized Weierstrass approximation theorem as given by Stone [Math. Mag. 21, 237–254 (1948); these Rev. 10, 255].

B. Yood (Ithaca, N. Y.).

Vulih, B. Z. On the concrete representation of partially ordered lineals. Doklady Akad. Nauk SSSR (N.S.) 78, 189–192 (1951). (Russian)

A real vector lattice ("lineal" in the author's terminology), for which every set bounded above has a least upper bound, admits a realization as a space of extended-real-valued continuous functions on a certain totally disconnected Hausdorff space [cf. Kantorovič, Vulih, and Pinsker, Funkcional'nyĭ analiz v poluuporyadočennyh prostranstvah, Moscow-Leningrad, 1950; these Rev. 12, 340]. Considering only a real vector lattice, subject, however, to other restrictions not clearly defined, the author states that such a representation exists in this case also. Extensions of spaces of continuous real functions to spaces of functions admitting infinities are also discussed.

E. Hewitt (Upsala).

Theory of Probability

de Finetti, Bruno. Sull'impostazione assiomatica del calcolo delle probabilità. Ann. Triestini. Sez. 2 (4) 3(19) (1949), 29–81 (1950).

The author gives a detailed discussion of the need (both psychological and mathematical) of some of the commonly accepted properties of probability, for example that probability measure is defined on a field of sets, and that it is completely (rather than simply) additive. *J. L. Doob.*

Lorenz, Paul. Der Schluss vom Teil aufs Ganze und der mathematischstatistische Begriff der Wahrscheinlichkeit. Veröffentlichungen Deutsch. Aktuarvereins 1, 25–43 (1951).

Addressed to statisticians, concerning the problem of drawing conclusions from a part (sample), applying to the whole (population). The various attempts to define and found probability are passed in review, with the well known difficulties indicated. A general descriptive approach based on frequencies in finite sets is suggested. There seems to be no really new contribution. *B. O. Koopman.*

von Schelling, Hermann. Distribution of the ordinal number of simultaneous events which last during a finite time. Ann. Math. Statistics 22, 452–455 (1951).

Given an urn containing white and black balls in proportions p and $1-p$, resp., one ball is drawn and returned during each unit of time, except that, if a white ball appears, play is interrupted for k units of time, then resumed. Given that at the nth time unit a white ball appears, the author derives the exact distribution, expected value and variance of the number m of white balls which have appeared since the beginning of play. Asymptotic expressions are given.

G. E. Noether (Boston, Mass.).

Kawata, Tatsuo, and Udagawa, M. A property of Poisson process. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, 10–15 (1951).

Let X_t be a homogeneous additive process taking only non-negative integers and $G(x)$ any proper distribution function; $F(x, t) = \sum_{k=0}^{\infty} \Pr(X_t = k) G^{k*}(x)$. Let $M_1(t)$, $M_2(t)$ be the first two moments of $F(x, t)$; m_1, m_2 those of $G(x)$. If $M_2(t)m_2^{-1} = M_1(t)m_1^{-1} + o(t)$ as $t \to 0$ then $M_1(t)m_1^{-1} = \lambda t$ and X_t is the Poisson process with parameter λ.

K. L. Chung (Ithaca, N. Y.).

Homma, Tsuruchiyo. On certain limiting distribution. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, 1–3 (1951).

The author treats the same problem as in a recent paper by Ugaheri [Ann. Inst. Statist. Math., Tokyo 1, 157–160 (1950); these Rev. 11, 731]. Numerous misprints.

K. L. Chung (Ithaca, N. Y.).

Ghosh, M. N. Convergence of random distribution functions. Bull. Calcutta Math. Soc. 42, 217–226 (1950).

The Fréchet-Shohat limit theorem is extended to the case where the sequence of distributions depends on a sequence of random variables. Convergence in probability or with probability one follows from the corresponding convergence of the moments, under certain conditions. Applications are given. *K. L. Chung* (Ithaca, N. Y.).

Feller, William. The asymptotic distribution of the range of sums of independent random variables. Ann. Math. Statistics 22, 427–432 (1951).

Let $\{S(t), 0 \leqq t < \infty\}$ be the random variables of a Brownian motion (Bachelier-Wiener) process, and let

$$R(t) = \operatorname*{Osc.}_{0 \leqq r \leqq t} S(r), \quad R^*(t) = \operatorname*{Osc.}_{0 \leqq r \leqq t} [S(r) - rS(t)/t].$$

The distribution of $R(t)$ has been found by Bachelier [Les nouvelles méthodes du calcul des probabilités, Gauthier-Villars, Paris, 1939]. The author expresses this distribution in a form more convenient for calculation, and also derives that of $R^*(t)$. The mean and variance of each distribution are then calculated. These quantities are derived in part to

be used as approximations of the corresponding quantities in the discrete parameter case. In the latter case $S(t)$ corresponds to the nth partial sum of mutually independent random variables with a common distribution function having a vanishing mean, and regular enough for the central limit theorem to be applicable. *J. L. Doob.*

Mann, Henry B. On the realization of stochastic processes by probability distributions in function spaces. Sankhyā 11, 3–8 (1951).

Let $\{x_t, t\epsilon T\}$ be a family of random variables (stochastic process) depending on the parameter t in the interval I. Let $\mathfrak{F}$ be the space of real-valued functions of $t\epsilon I$ and suppose that $\mathfrak{S}\subset\mathfrak{F}$. Then $\mathfrak{F}$ is a product space each of whose factors is the real line. Let x_t' be the tth coordinate function. Suppose that a probability measure of $\mathfrak{S}$ sets has been set up which makes each x_t' measurable and that the distribution of each finite aggregate of x_t''s is the same as that of the aggregate of x_t's with the same subscripts. Then the x_t' family of functions is said to realize the x_t family. Kolmogorov [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933] gave a general method of defining measures in function space and thereby showed that every stochastic process can be realized in this way if $\mathfrak{S}=\mathfrak{F}$. The author remarks that Kolmogorov's proof holds if $\mathfrak{S}$ is the class of functions of Baire class 2. He then states a condition on a stochastic process that it can be realized on the space of continuous functions. The condition (an immediate ϵ, δ continuity condition involving only finite parameter sets) is proved directly by the Kolmogorov method. It was proved earlier in a more general setting by the reviewer [Trans. Amer. Math. Soc. 42, 107–140 (1937)]. It is verified that the Brownian motion process satisfies the condition.

J. L. Doob (Urbana, Ill.).

Kemperman, J. H. B. Some remarks on the "random walk." Math. Centrum Amsterdam. Rapport ZW 1951-005, 13 pp. (1951). (Dutch)

Expository. *J. L. Doob* (Urbana, Ill.).

Tchen, Chan-Mou. Stochastic processes and dispersion of configurations of linked events. J. Research Nat. Bur. Standards 46, 480–488 (1951).

The author purports to give a rigorous derivation of Kolmogorov's diffusion equations. He fears that "the development of $p(s', r'; s, r)$ into a Taylor series will become badly convergent or even cease to be convergent at the limit $s'=s$," where $p(\cdots)$ is the transition probability density function which for $r=r'$, "must take necessarily an integrable infinite value, when s tends indefinitely to s'." In §III he ran into difficulties with inverse probabilities, of which an authentic definition has long existed, by requiring them to be the coefficients in the inverse of the linear transformation expressing the future absolute probabilities in terms of the initial and transition ones. *K. L. Chung.*

Mathematical Statistics

★Tables to Facilitate Sequential t-Tests. National Bureau of Standards Appl. Math. Ser., no. 7. U. S. Government Printing Office, Washington, D. C., 1951. xix+82 pp. \$.45.

An alternative is given to a sequential t-test introduced by Wald. A comparison of these tests based on an empirical sample is presented. Tables are given to facilitate the application of this alternative test. These tables may also be applied with minor modifications to the Wald test. These tables correspond to values of L, n and δ in the range $\pm L=2$, ln 19, 3, 4, ln 99, 5, 6, 7; $n=1(1)200$; $\delta=.1(.1)1.0(.2)2, 2.5$ where L, n and δ are defined as follows: If it is desired to test that the mean θ of a normal distribution with unknown variance σ is θ_0 against the alternative $|\theta-\theta_0|\geqq\delta\sigma$ so that the probabilities of error are no larger than α and β respectively, let $L_A=\ln[(1-\beta)/\alpha]$, $L_B=\ln[\beta/(1-\alpha)]$. If no decision is reached by sample size $n=200$, a procedure for truncating is suggested. *H. Chernoff* (Urbana, Ill.).

Masuyama, M., and Kuroiwa, Y. Table for the likelihood solutions of gamma distribution and its medical applications. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 1, 18–23 (1951).

Tables are presented to obtain the maximum likelihood estimates of the parameters a and p of the gamma distribution whose density is $(y/a)^{p-1}e^{-y/a}/a\Gamma(p)$ for $y\geqq 0$ and 0 for $y<0$. Formulae for the variances of the above estimates are given. Applications to the frequency characteristics of brain waves and the sedimentation rate during the period of normal pregnancy are given. *H. Chernoff.*

Fréchet, Maurice. Sur une application de la statistique mathématique à la biologie. Biometrics 7, 180–184 (1951).

A class of definitions of the typical member of a multidimensional population is discussed, with a view to improving on such concepts as that of "the average man".

L. J. Savage (Paris).

Ghosh, Birendranath. Some exponential forms for topographic correlation. Sankhyā 11, 29–36 (1951).

Stationary models of space correlation in two or more dimensions, with specific discussion of the degenerate case when the probability distribution is contained in a linear subspace. *H. Wold* (Uppsala).

Moran, P. A. P. Partial and multiple rank correlations. Biometrika 38, 26–32 (1951).

The author reviews various known facts concerning exact and asymptotic (for large samples) sampling distributions of Kendall's rank correlation coefficient t. He then proceeds to consider mean values of partial and multiple rank correlation coefficients defined from Kendall's t's in samples from a 3-variate population. The mean values are taken over all permutations of the values of the first variate in the sample, the values of the second and third variates in the sample being held fixed and all permutations being equally weighted. His results for partial rank correlation coefficients are very meager. In the case of the multiple rank correlation coefficient, the author finds an expression for the mean value of the square of the multiple rank correlation coefficient.

S. S. Wilks (Princeton, N. J.).

Picard, H. C. A note on the maximum value of kurtosis. Ann. Math. Statistics 22, 480–482 (1951).

The author shows that the maximum possible value for the kurtosis β_2 for populations of size N is $(N^2-3N+3)/(N-1)$. For the skewness β_1: max $\beta_1=(N-2)/(N-1)^{\frac{1}{2}}$; hence in the general inequality $\beta_2\geqq\beta_1^2+1$, the equality sign holds for the maxima. *A. M. Mood* (Santa Monica, Calif.).

Ramabhadran, V. K. A multivariate gamma-type distribution. Sankhyā 11, 45–46 (1951).

K. C. Cherian [J. Indian Math. Soc. (N.S.) 5, 133–144 (1941), pp. 136–137; these Rev. 3, 171] found the moment generating function (m.g.f.) of a bivariate gamma-type distribution function in which the marginal distributions are of the form $\lambda z e^{-z} dz$. The present author extends this result to 3 or more variates. He first lets u_1, u_2, u_3, u_4 be the sums of squares of different parts of independent standardized normal variables and then finds the joint frequency function of $x=u_1+u_2$, $y=u_2+u_3$, and $z=u_2+u_4$ and the corresponding m.g.f. He then gives the extensions to n such variables.

C. C. Craig (Ann Arbor, Mich.).

Pillai, K. C. S. On the distribution of an analogue of Student's t. Ann. Math. Statistics 22, 469–472 (1951).

Let $G=(\bar{x}-a)/w$, where $\bar{x}$ is the sample mean and w the sample range of a variable distributed normally with mean a and variance σ^2. The author expresses the probability function of G as an infinite series of terms involving Beta functions. The comparative efficiency of the G and Student's t test is tabled for various values of small sample sizes $n\leqq 20$ and the values of the confidence coefficient α, usually .05 or .01. For this range the efficiency of $G>.96$.

L. A. Aroian (Culver City, Calif.).

Cook, M. B. Bi-variate k-statistics and cumulants of their joint sampling distribution. Biometrika 38, 179–195 (1951).

The author derives formulas connecting κ_{rs}, the population cumulants, with μ'_{rs}, the population moments about the origin, and μ_{rs}, the population central moments, for $r+s\leqq 6$. Let (*) $\kappa[(\alpha\alpha')^r(\beta\beta')^s]$ represent the κ_{rs} of the joint distribution of $k_{\alpha\alpha'}$, and $k_{\beta\beta'}$, where $k_{\alpha\alpha'}$ and $k_{\beta\beta'}$ are the bivariate k-statistics. The author finds by two different methods, clearly explained, values of (*) for various combinations of r and s not readily summarized. The only joint distributions of bivariate statistics considered involve k_{11}, k_{21}, and k_{12}.

L. A. Aroian (Culver City, Calif.).

Ogasawara, Tôzirô, and Takahashi, Masayuki. Independence of quadratic quantities in a normal system. J. Sci. Hiroshima Univ. Ser. A. 15, 1–9 (1951).

The authors give concise proofs using matrix algebra for a rather complete list of theorems, including well known ones, concerning conditions under which quadratic forms in observations from a normal system obey χ^2-distributions, and under which two or more such forms are independently distributed. There are extensions of both kinds of results to multivariate quadratic forms. The theorems are too numerous and detailed to list here.

C. C. Craig.

Krishna Iyer, P. V. Further contributions to the theory of probability distributions of points on a line. II. J. Indian Soc. Agric. Statistics 3, 80–93 (1951).

The following extract from the paper summarizes its contents. "In part I of this series [same J. 2, 141–160 (1949); these Rev. 12, 271] the writer obtained the cumulants and the difference equations satisfied by the moment generating functions of a number of distributions arising from m points each possessing one of k characters or colours arranged at random on a line. The present paper deals with the difference equations of the probability generating functions (P.G.F.) of the distributions discussed in part I and also some other distributions which could not be discussed before. The P.G.F. is, as usual, defined by $\phi(m)=\sum\xi^r P(m,r)$, or $\phi(m)=\sum\xi_1{}^r\xi_2{}^s P(m,r,s)$, where $P(m,r)$ is the probability of having r variates of the distribution under consideration, while $P(m,r,s)$ refers to the joint distribution of two variates."

J. Wolfowitz (Ithaca, N. Y.).

Kawata, Tatsuo. Limit distributions of single order statistics. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, 4–9 (1951).

Let $X_{(k)}$ be the kth order statistic of n samples from a continuous population and let $F_{(k)}$ be its distribution function. If c is any fixed positive integer, all possible limits of $F_{(n-c)}(a_nx+b_n)$ as $n\to\infty$ are determined by using Gnedenko's results on the extreme order statistic. If $k/n\to 1$ and $n-k\to\infty$ it is shown for a class of distribution functions including all those belonging to the domain of attraction of all stable laws that the limit of $F_{(k)}(a_nx+b_n)$ is normal for suitable a_n and b_n.

K. L. Chung (Ithaca, N. Y.).

de Toledo Piza, Affonso P. Considerations on the geometric law. Trabajos Estadística 2, 79–104 (1951). (Portuguese. Spanish summary)

Several well-known theorems on sampling from a negative binomial universe (of which the discrete geometric universe is a special case) are proved without citation of previous work.

H. L. Seal (New York, N. Y.).

Oderfeld, J. On the dual aspect of sampling plans. Colloquium Math. 2, 89–97 (1951).

Simpson, Paul B. Note on the estimation of a bivariate distribution function. Ann. Math. Statistics 22, 476–478 (1951).

Let $F(x,y)$ be a continuous bivariate distribution function and $G_n(x,y)$ the empirical distribution function of a corresponding sample of n. It is shown that the distribution of $\sup|F(x,y)-G_n(x,y)|$ is not independent of the form of $F(x,y)$ (as the one-dimensional case might lead one to expect). It is suggested to reduce the two-dimensional case to the one-dimensional by considering only regions $y\leqq f(x)$, $x\leqq x_0$, where $f(x)$ is a monotonic function.

W. Feller.

Shenton, L. R. Efficiency of the method of moments and the Gram-Charlier Type A distribution. Biometrika 38, 58–73 (1951).

Inequalities are obtained for the efficiency of the method of moments for estimating the parameters of various distributions, including the type A distribution with four parameters.

D. Blackwell (Washington, D. C.).

Chernoff, Herman. A property of some type A regions. Ann. Math. Statistics 22, 472–474 (1951).

Neglecting sets of measure zero, if W_α and $W_{\alpha'}$ are the unique type A critical regions of size α and α' respectively $(1>\alpha>\alpha'>0)$ for testing a hypothesis, one might expect that $W_{\alpha'}\subset W_\alpha$. The author gives an example where, for all sufficiently small α, this property fails to hold.

J. Kiefer (Ithaca, N. Y.).

Barankin, Edward W. Concerning some inequalities in the theory of statistical estimation. Skand. Aktuarietidskr. 34, 35–40 (1951).

A simple proof of the Cramér-Rao inequality on the covariance matrix of a set of unbiased estimators of several parameters of a probability distribution, and of necessary and sufficient conditions on the estimators for equality to

be attained [e.g., same journal **29**, 85–94 (1946); these Rev. **8**, 163]. *J. Kiefer* (Ithaca, N. Y.).

Hoel, Paul G. Conditional expectation and the efficiency of estimates. Ann. Math. Statistics **22**, 299–301 (1951).

For estimating θ from a sample of size n from the distribution with density $f(x, \theta) = \exp[g(\theta) + h(\theta)r(x) + s(x)]$, there is, under certain regularity conditions, at most one unbiased estimate which is a function of the sufficient statistic $s = \sum r(x_i)$. Thus a device suggested by the reviewer [same Ann. **18**, 105–110 (1947); these Rev. **8**, 478] for improving an unbiased estimate leads only to this estimate.

D. Blackwell (Washington, D. C.).

Gronow, D. G. C. Test for the significance of the difference between means in two normal populations having unequal variances. Biometrika **38**, 252–256 (1951).

The author considers two ways of weighting the variances of the samples to obtain a variance for testing the significance of an observed difference in the means. For several sample sizes, significance levels, and comparative population variances he computes the power of the tests and the bias of assuming the distribution to be that of "t".

A. Blake (Buffalo, N. Y.).

van der Vaart, H. R. Directions for the use of Wilcoxon's test. Math. Centrum Amsterdam. Rapport S 32 (M4) 16 pp. (6 plates) (1950). (Dutch)

As the title indicates this is an exposition (with critical remarks, examples, and tables) of a test method in non-parametric statistical inference, for comparison of random variables, given first (in case of samples of equal size) by F. Wilcoxon [Biometrics Bull. **1**, 80–83 (1945)] and extended to samples of unequal size, by H. B. Mann and D. R. Whitney [Ann. Math. Statistics **18**, 50–60 (1947); these Rev. **9**, 151]. This author provides new tables to 7 places of decimals applicable through sample sizes up to 10 variates each. *A. Bennett* (Providence, R. I.).

Stein, C. M. A property of some tests of composite hypotheses. Ann. Math. Statistics **22**, 475–476 (1951).

The author constructs an example where the unique most powerful tests for testing a composite hypothesis against a simple alternative at levels of significance $\alpha_1 < \alpha_2$ are similar regions w_1 and w_2 such that w_1 is not contained in w_2.

E. L. Lehmann (Berkeley, Calif.).

Blomqvist, Nils. Some tests based on dichotomization. Ann. Math. Statistics **22**, 362–371 (1951).

The author presents nonparametric tests for testing row effects assuming no column effects in a two-way array of observations. Choose arbitrarily a percentile for each column and replace the observations by one or zero according as they do or do not exceed the percentile for that column. Various exact and asymptotic distributions are obtained for test criteria based on the row totals of these zeros and ones. In many important cases the tests reduce to simple chi-square tests. *A. M. Mood* (Santa Monica, Calif.).

Berger, Agnes. On uniformly consistent tests. Ann. Math. Statistics **22**, 289–293 (1951).

Call two sets $M = \{m\}$ and $M' = \{m'\}$ of measures distinguishable if for every $\epsilon > 0$ there is an integer n and a set B such that $m^n(B) > 1 - \epsilon$, $m'^n(B) < \epsilon$ for all $m \epsilon M$, $m' \epsilon M'$, where m^n denotes the n-fold product measure of m. A necessary and sufficient condition for M and M' to be distinguishable is given, which yields immediately that if M and M' are finite and disjoint, they are distinguishable. If $M = \{m_\theta\}$, $M' = \{m'_\tau\}$, where θ, τ vary over closed intervals of a finite dimensional Euclidean space, if M, M' are disjoint, and if, for every B, $m_\theta(B)$ and $m'_\tau(B)$ are continuous in θ, τ, then M and M' are disjoint. *D. Blackwell.*

Nandi, H. K. On analysis of variance test. Calcutta Statist. Assoc. Bull. **3**, 103–114 (1951).

Suggests a method for study of analysis of variance hypotheses using a combination of t and F tests. The method is less comprehensive and less well mathematically documented than Tukey's method [Biometrics **5**, 99–114 (1949); these Rev. **11**, 43]. See also Duncan [Virginia J. Sci. **2**, 171–189 (1951)]. *A. M. Mood* (Santa Monica, Calif.).

Frank, P., and Kiefer, J. Almost subminimax and biased minimax procedures. Ann. Math. Statistics **22**, 465–468 (1951).

Examples are given of decision problems in which the risk function of the minimax solution can be radically improved at "most" points, without being much worsened anywhere.

J. L. Hodges, Jr. (Chicago, Ill.).

Wolfowitz, J. On ϵ-complete classes of decision functions. Ann. Math. Statistics **22**, 461–465 (1951).

A class C of decision functions is "essentially ϵ-complete" if, for any δ_1 non-ϵ C, there exists $\delta_2 \epsilon C$ such that $r(F, \delta_2) \leqq r(F, \delta_1) + \epsilon$ for every distribution F. Under certain conditions (including boundedness of loss and compactness of the distributions and of the terminal decisions) there is shown to exist a finite essentially ϵ-complete class for each $\epsilon > 0$. *J. L. Hodges, Jr.* (Chicago, Ill.).

Wald, A., and Wolfowitz, J. Two methods of randomization in statistics and the theory of games. Ann. of Math. (2) **53**, 581–586 (1951).

In a sequential decision problem in which $x_1, x_2, \cdots$ are the possible observations and D is the set of terminal actions d, a non-randomized decision function f specifies for each n values of $x_1, \cdots, x_n$ which terminate sampling and, as a function of $x_1, \cdots, x_n$, the terminal action which is to be taken. Two types of randomized decision functions are considered: (1) probability measures over a suitable Borel field of subsets of the set F of possible f's and (2) the specification, for each n and each value of $x_1, \cdots, x_n$, of a probability measure over a suitable Borel field of subsets of D, according to which a terminal action is to be chosen if sampling stops. Under certain hypotheses on D, the two methods of randomization are equivalent in the sense that to any randomization of one type there corresponds a randomization of the other such that the probability distribution of (y, d), where y is the observed set of x's and d is the terminal action chosen, is the same for the two randomizations for all joint distributions of $x_1, x_2, \cdots$. *D. Blackwell.*

David, F. N., and Johnson, N. L. A method of investigating the effect of nonnormality and heterogeneity of variance on tests of the general linear hypothesis. Ann. Math. Statistics **22**, 382–392 (1951).

Under very general departures from the null hypothesis, formulae for the first four cumulants of $S - CT$ are derived, where C is a non-negative parameter, and S and T are the numerator and denominator of the F-test of a general linear hypothesis. Interesting specializations of these formulae are given. Some evidence is adduced that the distribu-

tion of $S-CT$ is reasonably well approximated by the Type IV distribution, agreeing with it in the first four cumulants. This is of course tantamount to recommending a certain approximation for the distribution of S/T, since $P(S/T>C)=P(S-CT>0)$. *L. J. Savage* (Paris).

Steel, Robert G. D. Minimum generalized variance for a set of linear functions. Ann. Math. Statistics **22**, 456–460 (1951).

Let X_{ij} $(j=1, \cdots, n_i, i=1, \cdots, k)$ be random variables with given covariance matrix. The author derives a system of equations that must be satisfied by the coefficients a_{ij} for which the variables $Y_i=\sum a_{ij}X_{ij}$ have minimum generalized variance when each has unit variance. The a_{ij} are determined explicitly for a simple example. *E. L. Lehmann.*

Kendall, M. G. Regression, structure and functional relationship. I. Biometrika **38**, 11–25 (1951).

The logic of inference in regression analysis is discussed, with a stress on two types of approach, viz. when the dependent and independent variables are assumed to have a joint probability distribution (the Galton-Yule specification), and when the dependent variable is assumed to have a probability distribution which involves the independent variables as parameters (the Fisher-Bartlett specification). *H. Wold* (Uppsala).

Durbin, J., and Kendall, M. G. The geometry of estimation. Biometrika **38**, 150–158 (1951).

Euclidean geometric treatment of the multiparameter linear regression problem from the Markoffian viewpoint. There is an application, which the reviewer is unable to understand, to the method of maximum likelihood. *L. J. Savage* (Paris).

Durbin, J., and Watson, G. S. Testing for serial correlation in least squares regression. II. Biometrika **38**, 159–178 (1951).

This article presents some applications of the theory presented in part I [Biometrika **37**, 409–428 (1950); these Rev. **12**, 512]. This theory concerned the development of a test criterion for testing the error terms of a regression model for serial correlation. Upper and lower bounds to the 5%, 2.5% and 1% points of the distribution function of this criterion are presented for 1 to 5 fixed variates in the regression model and for 15 to 100 observations. An approximate procedure of testing is also outlined when the bounds test is inconclusive. The following problems are considered: regression on two fixed variates, analysis of variance with a two-way classification, and polynomial regression. Some corrections to errors in part I are also included. *R. L. Anderson.*

Watson, G. S., and Durbin, J. Exact tests of serial correlation using noncircular statistics. Ann. Math. Statistics **22**, 446–451 (1951).

This note presents an exact non-circular test of the existence of serial correlation in a series of n observations. Tables of the 5% significance points are presented for $n=10(2)22$ if the mean is known a priori, and for $n=12(2)30$ if the mean is unknown. The results were obtained by omitting one or two of the middle terms in the sum of the successive products and using distributions derived by the reviewer [same Ann. **13**, 1–13 (1942); these Rev. **4**, 22]. *R. L. Anderson.*

Anderson, T. W. Estimating linear restrictions on regression coefficients for multivariate normal distributions. Ann. Math. Statistics **22**, 327–351 (1951).

The author considers a complex of statistical variates in which a number of the variates are assumed to be representable by a linear regression upon the remaining variates. If now there exist linear relations between the elements of the matrix of regression coefficients, these may be interpreted as linear stochastic relations between the regressors. For instance, if the endogenous variates of an economic system are represented by a linear regression upon the exogenous variates (reduced form) the stochastic relations mentioned above are simply the "structural equations" of the economic model. The author's central problem is to determine the existence of such relations, and then to estimate them.

The maximum likelihood estimates of the linear relations are shown to be the latent vectors of a certain matrix. The latent roots of this matrix also provide optimum sample functions for testing hypotheses on the rank of the matrix of regression coefficients. A test for the admissibility of a given set of relations is deduced, and a confidence region defined by all such relations not rejected by this test. However, quite a number of conditions must be imposed upon the relations in order to attain determinacy, and the shape of the confidence region is dependent upon these rather arbitrary conditions. The author indicates that all of the confidence regions here deduced are asymptotically consistent, but the question of deciding between them is for the time being left open. *P. Whittle* (Uppsala).

Drion, E. F. Estimation of the parameters of a straight line and of the variances of the variables, if they are both subject to error. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 256–260 (1951).

Let x and y be random variables with the structure, $x_i=\xi_i+u_i$, $y_i=\alpha+\beta\xi_i+v_i$ $(i=1, 2, \cdots, n)$, in which α, β, and the ξ_i are unknown finite parameters, while u_i and v_i are independent random variables with zero first and third moments and finite second, fourth, fifth, and sixth order moments. Neither the distribution of the u_i's nor that of the v_i's depends on i or on ξ_i. Given n pairs of independent observed values (x_i, y_i), $i=1, 2, \cdots, n$ the author shows how to find moment estimates of α, β, the variances of x and y and the first three moments of the ξ_i, which he shows are consistent. *C. C. Craig* (Ann Arbor, Mich.).

Freeman, G. H., and Halton, J. H. Note on an exact treatment of contingency, goodness of fit and other problems of significance. Biometrika **38**, 141–149 (1951).

Presents a computational procedure for determining exact significance levels of χ^2 when applied to contingency tables. *A. M. Mood* (Santa Monica, Calif.).

Wold, Herman O. A. Stationary time series. Trabajos Estadística **2**, 3–74 (1951). (Spanish)

Expository treatment, based on lectures given in Madrid in 1949 and in Lucknow in 1950. The main emphasis is on regression problems. *J. L. Doob* (Urbana, Ill.).

Robbins, Herbert, and Monro, Sutton. A stochastic approximation method. Ann. Math. Statistics **22**, 400–407 (1951).

Let $M(x)$ denote the expected value at level x of the response to a given experiment. $M(x)$ is unknown to the ex-

perimenter and the authors consider the problem of finding the solution $x=\theta$ of the equation $M(x)=\alpha$ (where α is a given constant) on the basis of successive experiments at levels $x_1, \cdots, x_n, \cdots$, in such a way that x_n tends in probability to θ. The authors consider the following method of successively approximating θ. Define $x_{n+1}=x_n+a_n(\alpha-y_n)$ where x_1 is arbitrary, y_n is a random variable such that $\Pr(y_n\leqq y|x_n)=H(y|x_n)$ and $M(x)=\int_{-\infty}^{\infty}ydH(y|x)$, and $\{a_n\}$ is a fixed sequence of positive constants. Then it is proved that if the a_n are such that $\sum_1^\infty a_n^2<\infty$ and $\sum_{n=2}^\infty a_n/(a_1+\cdots+a_{n-1})=\infty$ and if $M(x)$ is nondecreasing, $M(\theta)=\alpha$ and $M'(\theta)>0$, then $\lim_{n\to\infty} E(x_n-\theta)^2=0$ and this implies convergence in probability of x_n to θ. These results are applied to a problem of estimation of a quantile using response, no-response data and the possibilities of using this method to solve rather general regression problems are described. *R. P. Peterson* (Seattle, Wash.).

Banerjee, K. S. Some observations on the practical aspects of weighing designs. Biometrika 38, 248–251 (1951).

Illustrates the use of balanced incomplete block designs for the spring balance. *A. M. Mood.*

***Statistical Methodology Reviews, 1941–1950. Edited by Oscar Krisen Buros.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1951. x+457 pp. $7.00.

A collection of reviews or excerpts from reviews of over 300 books or pamphlets on statistics or its applications. Only books written in English and published or reviewed during the period 1941–1950 are included.

Mathematical Economics

***Samuelson, Paul A. Abstract of a theorem concerning substitutability in open Leontief models.** Activity Analysis of Production and Allocation, pp. 142–146. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

A Leontief model may be described as follows: A collection of n goods is to be produced by n industries, the ith industry producing a positive amount of the ith good and non-positive amounts of the other $n-1$ goods. A fixed amount of labor (the $(n+1)$st good) is available to the n industries and it is assumed that the amount of each good produced (or consumed) by an industry in a given process is proportional to the amount of labor consumed. The model is said to be producing "efficiently" if it is producing in such a way that it is impossible to increase the output of any good without decreasing the output of some other good. There will in general be infinitely many modes of efficient production depending on the production processes in the individual industries. The author however proves the following theorem: In each of the n industries there exists a single process such that all possible modes of efficient production are obtainable by suitably distributing labor among these n processes. For the proof it is assumed that each output depends differentiably on the inputs, and the method of Lagrangean multipliers is used. *D. Gale.*

***Koopmans, Tjalling C. Alternative proof of the substitution theorem for Leontief models in the case of three industries.** Activity Analysis of Production and Allocation, pp. 147–154. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The theorem stated in the previous review is here proved for the special case of three industries but without any assumptions of differentiability of production functions. The proof appears to depend on reasoning from a diagram although no argument is given to show that the particular diagram used represents the general case. *D. Gale.*

***Arrow, Kenneth J. Alternative proof of the substitution theorem for Leontief models in the general case.** Activity Analysis of Production and Allocation, pp. 155–164. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The theorem of the two preceding reviews is proved for the general case (n industries) without using any assumptions of differentiability. The theorem here is given a purely geometric formulation as follows. If a and b are points of n-space we write $a\geqq b$ to mean $a_i\geqq b_i$ for all i, and $a_i>b_i$ for some i. A point a of a convex set S in n-space is called an "efficient point" if $b\geqq a$ implies b non-$\epsilon\, S$. Theorem: For each $j=1, 2, \cdots, n$ let S_j be a convex set in n-space such that if $a\epsilon S_j$ then $a_i\leqq 0$ for $i\neq j$. Let S be the intersection of the convex hull of the union of $S_1, S_2, \cdots, S_n$ with the non-negative orthant ($\{a|a\geqq 0\}$). If S is compact and contains at least one positive element (element a such that $a_i>0$ for all i) then the set of efficient points of S is the intersection of an $(n-1)$-dimensional hyperplane with the non-negative orthant, the direction coefficients of whose outward normal are all positive. *D. Gale* (Providence, R. I.).

***Georgescu-Roegen, Nicholas. Some properties of a generalized Leontief model.** Activity Analysis of Production and Allocation, pp. 165–173. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

A Leontief model consists of a collection of processes which may be used to produce n commodities $G_1, G_2, \cdots, G_n$, with the following restrictions: (1) each commodity can be produced by at least one process; (2) no process can produce positive amounts of more than one commodity; (3) each process consumes a positive amount of labor, the $(n+1)$st commodity, denoted by G_{n+1}; (4) the quantity of output of a commodity in any process is proportional to the quantity of input of commodities. The author states, without giving proofs, a list of thirteen theorems, plus numerous corollaries concerning such models. Typical is the following. One says that a given bill of goods can be produced by labor alone, if it is possible to produce this bill of goods so that there is no net decrease in any of the commodities $G_1, G_2, \cdots, G_n$. A complete bill of goods is one in which the amounts of each G_i are positive. Theorem: If it is possible to produce one complete bill of goods by labor alone, then any bill of goods can be produced by labor alone. *D. Gale.*

Bílý, Josef. Sinking funds. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1948, no. 6, 18 pp. (1949). (English. Czech summary)

This paper deals with the problem of the formation of sinking funds with variable final amount. Suppose Sc_k is

the final amount, estimated at the end of k intervals and satisfying the condition that $c_0=1$. Various solutions are considered, but the main concern here is with the method whereby the difference between Sc_k and the fund actually saved at the end of k periods is distributed among the remaining $n-k$ periods.

For the discontinuous solution let the instalments paid into the fund be (1) $\gamma_k=\alpha_k\beta_k$ $(k=1, 2, \cdots, n)$, where α_k denotes the size of instalments assuming that the final amount is constant and β_k is a factor depending on c_k. Equation (1) may be written as (2) $\gamma_k=\alpha e_k\beta_k$, where the e_k determine the instalment scheme. If F_{k-1} is the amount of the fund at the end of $(k-1)$ periods, $F_{k-1}(1+i)^{n-k+1}$ is its value at the end of n periods due to interest. At the end of k periods the final total is estimated as Sc_k, so it will be required to provide

$$(3)\qquad Sc_k-F_{k-1}(1+i)^{n-k+1}=\alpha\beta_k s_{\overline{n-k+1}|}(e_k, e_{k+1}, \cdots, e_n)$$

in $(n-k+1)$ periods. If at the end of each period c_k is assumed to be fixed until the end of the transaction, we have

$$(4)\qquad \alpha\beta_k=\frac{Sc_k}{s_{\overline{n-k+1}|}(e_k, e_{k+1}, \cdots, e_n)}-\frac{F_{k-1}}{a_{\overline{n-k+1}|}(e_k, e_{k+1}, \cdots, e_n)},$$

where, in general, $s_{\overline{n}|}(e_1, \cdots, e_n)=\sum_{k=1}^{n}e_k(1+i)^{n-k}$ and $a_{\overline{n}|}(e_1, \cdots, e_n)=\sum_{k=1}^{n}e_k(1+i)^{-k}$.

Since the relation $F_{k-1}(1+i)+\gamma_k=F_k$ holds, we obtain the recurrent formula

$$(5)\qquad F_k=F_{k-1}\frac{a_{\overline{n-k}|}(e_{k+1}, \cdots, e_n)}{a_{\overline{n-k+1}|}(e_k, \cdots, e_n)}+\frac{Sc_ke_k}{s_{\overline{n-k+1}|}(e_k, \cdots, e_n)}$$

with initial condition $F_0=0$.

By successive use of (5),

$$(6)\qquad F_k=Sa_{\overline{n-k}|}(e_{k+1}, \cdots, e_n)\times\sum_{j=1}^{k-1}\frac{c_je_j}{s_{\overline{n-j+1}|}(e_j, \cdots, e_n)a_{\overline{n-j}|}(e_{j+1}, \cdots, e_n)}+\frac{Sc_ke_k}{s_{\overline{n-k+1}|}(e_k, \cdots, e_n)}.$$

Equations (4) and (6) may be reduced to special cases depending on the assumed behavior of c_k and e_k. The continuous solution is given with the generalization that the intensity of interest, defined by the relation

$$\delta(t)=[1/K(t)]\cdot dK(t)/dt,$$

where $K(0)$ is initial capital at zero time, varies with time. The growth of capital from t_1 to t_2, say, for intensity $\delta(t)$ measured from $t=0$ is given by

$$K(t_2)=K(t_1)\exp\int_{t_1}^{t_2}\delta(\tau)d\tau=K(t_1)\exp[\varphi(t_2)-\varphi(t_1)].$$

The solution is now everywhere analogous to the discontinuous solution. Finally, generalizations of the continuous solution to cover cases where the amount already paid into the fund at a certain time point is altered by a change in c and where the operation of the sinking fund is dependent on human life as in social and private insurance are considered.

M. P. Stolts (Providence, R. I.).

Houthakker, H. S. Revealed preference and the utility function. Economica N. S. 17, 159–174 (1950).

The author shows how a generalization of Samuelson's axiom of revealed preference can be necessary and sufficient for consumers' choice to be explainable on the basis of an integrable utility function. The axiom reads: "If $X^0, X^1, X^2, \cdots, X^T$ is a sequence of batches of goods such that each batch is bought at prices $P^0, P^1, P^2, \cdots, P^T$ respectively, and if at least two of these batches are different, and if the cost $P^{t-1}X^t$ [inner product] of each batch X^t at prices P^{t-1} is not greater than the cost $P^{t-1}X^{t-1}$ of the preceding batch in the sequence X^{t-1} at the same prices, then the cost P^TX^T of the last batch X^T at prices P^T is less than the cost P^TX^0 of the first batch X^0 at the same prices." The axiom amounts to saying that if the quantity of goods purchased is compared by ordinary index-number methods with that at the preceding time period and is found to be increasing all the time, then the last batch of goods will not be found to be less than the first. The axiom is clearly valid if the choice made by the consumer is made so as to maximize his utility function subject to a budget restraint; it is shown that the converse is also true, i.e., that if the axiom is satisfied, then there is a function over the space of batches of goods such that the choice made under any budget restraint is the one which maximizes the function subject to the restraint that the cost of the batch not exceed the prescribed budget. Essentially the same theorem has been established by J. Ville [Ann. Univ. Lyon. Sect. A. (3) 9, 32–39 (1946); these Rev. 8, 396] by a different proof.

K. J. Arrow (Stanford University, Calif.).

TOPOLOGY

Cantoni, Riccardo. Ricerche sulle reti con vertici tripli. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 12(81), 55–70 (1948).

The most important contribution of the present paper is the concept of "snake" (serpe) and a method for transforming a snake into another snake. A snake, in any cubic map of n simply connected regions, is a simple open curve (without double points) consisting only of boundary points of regions, with one end (the head) at a vertex H and the other end (the tail) at another vertex T, and such that it passes just once through each of the other $2n-6$ vertices of the map. The method of transforming the snake is to add to the snake a boundary arc joining T to a vertex J (where J is necessarily on the snake but is chosen distinct from H), and then subtracting from the snake that boundary arc $T'J$ such that T' lies on the snake between J and T. The result is a new snake with the same head H but with a different tail, T'. The author makes the experimentally very plausible hypothesis that every maximal set of snakes that can be changed into each other by a repetition of this process contains at least one in which the head and tail are connected by a boundary arc (not belonging to the snake). With this hypothesis an inductive proof can be furnished for the conjectured theorem that there always exists a simple closed curve, consisting only of boundary points of regions and passing just once through each of the $2n-4$ vertices of the map. As is well known this would imply the four color theorem. When once we have a snake drawn on a map, we have a mechanism which, if this hypothesis is correct, leads to a sure way of coloring the map without false moves. The paper contains a number of other minor items, one of which is a proof of a theorem essentially equivalent to Petersen's theorem.

D. C. Lewis (Baltimore, Md.).

Senior, James K. Partitions and their representative graphs. Amer. J. Math. 73, 663–689 (1951).

If G is any finite graph let $P(n, G)$ denote the number of vertices of G of degree n. If $P=P(n)$ is any function of the positive integer n let $ZCL(P)$ be the number of non-isomorphic graphs G, connected and without loops, such that $P(n, G)=P(n)$ for each n. The author sees little hope of making a general determination of $ZCL(P)$ in terms of P. He does however make a complete classification of those functions P which satisfy $ZCL(P)=0$ or $ZCL(P)=1$. $ZCL(P)$ may be regarded as the number of different molecules having just $P(n)$ atoms of valency n, only one element of each valency being involved. The author generalizes his result to the case in which two or more elements of the same valency may occur, each with a fixed number of atoms.

W. T. Tutte (Toronto, Ont.).

Nöbeling, Georg. Zur Theorie der topologischen Räume. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1950, 131–132 (1951).

The author announces that in a book of his, "Analytische Topologie" (to appear in 1953), the following axioms will form the starting point of his presentation of the theory of topological spaces and topological lattices: $A\subset A^-$; if $A\subset B$ then $A^-\subset B^-$. *R. Arens* (Los Angeles, Calif.).

Hanai, Sitiro. On N. Matsuyama's closure operators on general neighbourhood spaces. Tôhoku Math. J. (2) 3, 45–47 (1951).

N. Matsuyama [same J. (2) 1, 22–45 (1949); these Rev. 11, 165] presented a method of defining a "φ-closure operator" ($A\rightarrow A^\varphi$) when there was given a set-valued set function $A\rightarrow\varphi A$ in a neighborhood space. In the present paper, the mapping from the lattice of all φ to the lattice of all A^φ (A being fixed) is shown to be an open continuous dual homomorphism, using the star-topology. *R. Arens.*

Choquet, Gustave. Difficultés d'une théorie de la catégorie dans les espaces topologiques quelconques. C. R. Acad. Sci. Paris 232, 2281–2283 (1951).

The following facts are pointed out or proved: (1) any metric space is complete in some uniform structure, (2) spaces with totally ordered uniform structure are either metrizable or no countable set has a limit point, (3) using the continuum hypothesis, one can construct a compact space which is the union of a nested system of nowhere dense closed sets, (4) in any Hausdorff space X any set A containing the isolated points of X is the set of points of continuity of some function f with values in a compact space. These facts more or less destroy each of four conceivable attempts to generalize Baire's density theorem. *R. Arens.*

Klee, V. L., Jr. Some characterizations of compactness. Amer. Math. Monthly 58, 389–393 (1951).

The author shows how to imbed any metric space in the boundary of a symmetric convex open set in its space of continuous bounded real-valued functions. Using this construction, he gives short proofs of results due to H. E. Vaughan [Bull. Amer. Math. Soc. 44, 557–561 (1938)] such as, if X is metrizable but not compact, it has a bounded metric which is neither totally bounded nor complete. There are also some new results, for example: if X is metrizable but not compact, there is a metric ρ such that for each x, sup $\rho(x, y)$, y ranging over all X, is less than the diameter of X. *R. Arens* (Los Angeles, Calif.).

Arnold, B. H. Topologies defined by bounded sets. Duke Math. J. 18, 631–642 (1951).

Let S be a real vector space and let $\mathfrak{B}$ be a family of subsets of S which contains all finite subsets and is closed with respect to the formation of finite set theoretical unions, subsets, convex hulls and scalar multiples. If, in addition, no ray through the origin is contained in any member of $\mathfrak{B}$, then $\mathfrak{B}$ is said to define a notion of boundedness in S. The subset G is said to be open with respect to $\mathfrak{B}$ if $g\epsilon G$ and $B\epsilon\mathfrak{B}$ imply that $g+\lambda B\subseteq G$ for some positive real number λ. Elementary properties of these notions are studied. For example: (a) The topology defined by the sets open with respect to $\mathfrak{B}$ makes S a T_1 space but not necessarily a topological linear space. (b) If S is finite-dimensional then all $\mathfrak{B}$'s coincide.

G. W. Mackey (Cambridge Mass.).

Hewitt, Edwin. Linear functionals on spaces of continuous functions. Fund. Math. 37, 161–189 (1950).

Let $C(X, R)$ denote the algebra of all continuous functions (unbounded as well as bounded) on a completely regular space X with values in R, the real number system. Let $P(X)$ be the class of all sets of the form $P(\varphi)=E[x; x\epsilon X, \varphi(x)>0]$ for $\varphi\epsilon C(X, R)$ and let $\overline{P(X)}$ be the smallest σ-field of subsets of X containing $P(X)$. A non-infinite set function completely additive on $\overline{P(X)}$ is called a Baire measure. A linear functional on $C(X, R)$ is called bounded if it takes bounded sets of functions into bounded sets of R.

The first part of the paper is mainly devoted to showing that to each bounded linear functional I on $C(X, R)$ there corresponds a Baire measure γ such that every $f\epsilon C(X, R)$ is bounded except on a set of γ-measure zero and such that $I(f)=\int_X f(x)d\gamma$. This γ is obtained from an outer measure γ^* defined in terms of I and is a regular Baire measure. The author connects this theory with the theory of Q-spaces developed by him previously [Trans. Amer. Math. Soc. 64, 45–99 (1948); these Rev. 10, 126]. He characterizes Q-spaces among the completely regular spaces by a simple property of the Baire measure. Also if X is a Q-space, for an integral representation of a positive linear functional I, it suffices to confine oneself to a suitably chosen compact subset of X.

Next the author considers various topologies for $C(X, R)$ and gives properties of linear functionals continuous in these topologies. One of his results here shows again that the situation is neater for Q-spaces. If X is a Q-space then I is continuous in the k-topology if and only if I is bounded whereas if X is not a Q-space there is always a bounded I which is not continuous in that topology. Several topologies are considered for the space B of all bounded linear functionals on $C(X, R)$. By a suitable definition of bounded sets in $C(X, R)$ a topology can be defined for B in the manner of Arens [Duke Math. J. 14, 787–794 (1947); these Rev. 9, 241]. For example, with the k-topology the bounded sets are the subsets S of $C(X, R)$ for which $\sup_{f\epsilon S}(\sup_{x\epsilon K}|f(x)|)<\infty$ for every compact subset K of X. It is shown that, in the associated topology for B, if X is a Q-space, B is a complete partially ordered locally convex topological linear space. The author also considers two types of weak topologies for B and connects them with properties of the ring-homomorphisms in B. The author also shows how the theory of Markov [Mat. Sbornik N.S. 4, 165–190 (1938)] may be reduced to the present theory.

B. Yood (Ithaca, N. Y.).

Bertolini, Fernando. Applicazione di un noto criterio generale di compattezza allo spazio lagrangiano delle funzioni continue. Boll. Un. Mat. Ital. (3) **6**, 107–110 (1951).

Suppose A and B are Hausdorff spaces, B^A is the space of continuous maps of A into B (with the compact-open topology), and C is a subset of B^A. In recent years, several authors have found criteria for the compactness of C which reduce in the classical case (where A is a bounded set in E^n and B is E^1) to equiuniform continuity and uniform boundedness, thus generalizing the Ascoli-Arzela theorem. Arens [Ann. of Math. (2) **47**, 480–495 (1946); these Rev. **8**, 165] treats the case in which A and B are uniform spaces, and the discussion of Gale [Proc. Amer. Math. Soc. **1**, 303–308 (1950); these Rev. **12**, 119] is still more general. (See also Myers [Ann. of Math. (2) **47**, 496–502 (1946); these Rev. **8**, 165].) In the paper under review, the author proves the theorem in its classical form, using the fact that in a complete metric space, total boundedness implies compactness.

V. L. Klee, Jr. (Princeton, N. J.).

Estill, Mary Ellen. Concerning abstract spaces. Duke Math. J. **17**, 317–327 (1950).

A study of the implications, and relations to various alternatives, of the system A consisting of Axioms 0 and 1 of R. L. Moore's "Foundations of Point Set Theory" [Amer. Math. Soc. Colloquium Publ., v. 13, New York, 1932]. Moore has previously shown [Proc. Nat. Acad. Sci. U. S. A. **28**, 56–58 (1942); these Rev. **3**, 136] that a model of A is separable if it contains no uncountable set of disjoint domains. It is shown that this no longer holds if part (4) of Axiom 1 is deleted, however. The systems alternative to A are obtained by modifying Axiom 1, chiefly part (4) thereof.

R. L. Wilder (Ann Arbor, Mich.).

Estill, Mary Ellen. Separation in non-separable spaces. Duke Math. J. **18**, 623–629 (1951).

This paper is concerned with spaces that satisfy Axiom 0 and Axiom 1 (except for condition 4) of R. L. Moore [Foundations of Point Set Theory, Amer. Math. Soc. Colloquium Publ., v. 13, New York, 1932], are not separable and do not contain uncountably many mutually exclusive domains. Let a space satisfying these conditions be called an E space. The author [see the preceding review] has exhibited an example of an E space and R. L. Moore [Proc. Nat. Acad. Sci. U. S. A. **28**, 56–58 (1942); these Rev. **3**, 136] has shown that no E space exists as a subspace of a space that satisfies all of Axiom 1. In the present paper the author constructs a locally connected E space and shows that for no locally connected E space is it true that whenever A and B are points of E and R is a region containing A, then R contains a separable or finite subset separating A and B. However, a connected E space is constructed which has the property that if A and B are points of E and R is a region containing A, there is a point lying in R which separates A and B.

W. R. Utz (Columbia, Mo.).

Cassina, Ugo. Il concetto di linea piana e la curva di Peano. Rivista Mat. Univ. Parma **1**, 275–292 (1950).

Nel presente lavoro, storico divulgativo, premessa una rapida sintesi storica del modo con cui i matematici, prima della scoperta della curva di Peano, avevano creduto di formulare razionalmente i concetti intuitivi di linea, di superficie e di solido, riporterò la definizione analitica data da G. Peano della sua curva e la illustrerò graficamente in due modi, di cui il secondo è forse nuovo. (Extract from paper.)

R. H. Fox (Utrecht).

Zarankiewicz, Kasimierz. Images réciproques de fonctions continues univoques et le principe de Dirichlet. Soc. Sci. Lett. Varsovie C. R. Cl. III. Sci. Math. Phys. **41** (1948), 1–7 (1950). (French. Polish summary)

Let f map the unit circle onto the Peano space C. If the complement in C of a point x has n components then $f^{-1}(x)$ has cardinal at least n [Nöbeling, Fund. Math. **20**, 30–46 (1933)]. The author proves this and a related local result. "Le principe de Dirichlet" is not given its customary meaning.

A. D. Wallace (New Orleans, La.).

Cartwright, M. L., and Littlewood, J. E. Some fixed point theorems. With appendix by H. D. Ursell. Ann. of Math. (2) **54**, 1–37 (1951).

The paper is concerned with fixed-point theorems for plane continua which are acyclic but otherwise unrestricted, and are therefore beyond the reach of existing algebraic fixed-point theory. The mappings are required to be extensible to topological mappings of the whole plane on to itself. The need for theorems of this generality arises naturally from the consideration of the differential equation $\ddot{x}+f(x)\dot{x}+g(x)=p(t)$, where p has the period 1 and f, g and p are "good" enough to ensure existence and uniqueness of solutions everywhere. Given any point (x, y) let $x(t)$ be the solution such that $x(0)=x$, $\dot{x}(0)=y$, and let $T(x, y)$ be the point $(x(1), \dot{x}(1))$. Then T is a homeomorphism of R^2 whose fixed points correspond to periodic solutions of the equation. It was shewn by Levinson [J. Math. Physics **22**, 41–48 (1943); Ann of Math. (2) **45**, 723–737 (1944); these Rev. **5**, 66; **6**, 173] that if, for large x, $f\geqq\alpha>0$ and $g(x)/x\geqq\beta>0$, there exists a Jordan domain D such that $T\bar{D}\subseteq D$ and hence $\bigcap T^n\bar{D}$ is an acyclic continuum F invariant under T. The classical Brouwer theorem applied to $T^n\bar{D}$ shews that F contains a fixed point of T; but for the study of the differential equation it is fixed points on the frontier $\mathfrak{F}F$ that are interesting, and it is the search for these points that leads to the theorems of the present paper. Very simple functions f, g, p (e.g. van der Pol's equation $\ddot{x}+k(x^2-1)\dot{x}+\omega^2x=bk\cos 2\pi t$) lead to sets F whose frontiers are not locally connected, and perhaps contain indecomposable continua (see the end of this review).

Let T be a topological mapping of R^2 on to itself and I an acyclic continuum such that $TI=I$. The authors consider the mapping of the prime ends of R^2-I on to each other by T. A central problem is to find conditions under which the point-set locus, P, of a fixed prime end $\mathcal{P}$ ($T\mathcal{P}=\mathcal{P}$) necessarily contains a fixed point of T. That this is not always so is seen by taking I to be the union of $|z|\leqq 1$ and an arc spiralling in to the circle. A fixed prime end defined by the sequence of domains D_m is, by definition, stable under T if, given m, there exists $n_0(m)$ such that $T^n\bar{D}_m\subseteq D_{m+1}$ if $n>n_0$; it is unstable for T if it is stable for T^{-1}; and non-stable if it is neither stable nor unstable. Theorem 1 states that every non-stable fixed prime end contains a fixed point; whence (Theorem 3) if I is of the special form $\bigcap T^n\bar{D}$, where D is a Jordan domain such that $T\bar{D}\subseteq D$, and if there is a fixed prime end, then there is a fixed point on $\mathfrak{F}I$. The main existence theorem (Theorem 5) states that any acyclic continuum I which is invariant under a topological mapping T of the whole plane on to itself contains a fixed point. The general line of the argument is as follows. It is first shewn (Theorem 4) that if there is no fixed prime end there is a

fixed point on I. This, with Theorem 1, shews that, if there is no fixed point on I, there is a stable or unstable prime end, and it is easily seen that there must be at least one of each, terminating an invariant "line end" (the image, under the usual Carathéodory mapping, of an invariant arc of $|z|=1$ terminated by a stable and an unstable fixed point). This enables a subcontinuum I' of I, and a simple arc C in R^2-I, to be constructed such that (1) $C^*=\bigcup_{-\infty}^{\infty}T^nC$ is an open arc, (2) the closures of the two halves, $\bigcup_0^\infty T^nC$ and $\bigcup_0^\infty T^{-n}C$, of C^* both contain $\mathfrak{F}I'$, and (3) there is no fixed point of T in $\overline{D^*}$, where D^* is the bounded domain between C^* and I'. Thus, given $\eta>0$, an arc C_{ab} of C^*, containing C, and a simple arc L_{ab}, lying within η of I' and of diameter $<\eta$, can be found, such that $C_{ab}\cup L_{ab}$ is a simple closed curve. It is now shewn that if η is small enough, first (Lemmas 29, 30, 31) the inner domain, D, of $C_{ab}\cup L_{ab}$ lies in so close a neighbourhood of $\overline{D^*}\cup I'$ that it can contain no fixed point, and secondly (Lemma 32) using the fact that T maps $C_{ab}\cup L_{ab}$ mainly on to itself, that D must contain a fixed point. This contradiction shews that I contains a fixed point.

Finally the converse problem is considered: what can be inferred about the mapping of prime ends from the existence of a fixed point on $\mathfrak{F}I$? The following theorem is proved by means of Poincaré's rotation numbers (I and T being as before). Theorem 6. If there exists a fixed point p on $\mathfrak{F}I$, but no fixed prime end, if $\mathcal{P}$ is a prime end and $p\in P$, and if C is an open arc in R^2-I converging to $\mathcal{P}$, and such that $p\in\bar{C}$, then for some positive N all fixed points in I are in the complement of the unbounded residual domain of $I\cap\bigcup_0^N T^n\bar{C}$. In an appendix by H. D. Ursell a shorter proof is given, with a discussion of possible values of N.

Examples are discussed at several points of the paper. In particular, a result (Theorem 7) is deduced from Theorem 6 which, if certain conjectures made by the authors [J. London Math. Soc. 20, 180–189; these Rev. 8, 68; and unpublished work] about the set F (above) of van der Pol's equation were confirmed, would shew that $\mathfrak{F}F$ contains an indecomposable continuum.

[On p. 2, l. 4 from below, "not stable" should be "non-stable." In the definition of "external stability," p. 14, the D_n are presumably to satisfy $D_{n+1}=TD_n$. $J(\eta, \infty)$ should be $J(\eta, N)$ wherever it occurs. In the proof of Lemma 32, η must be so small that neither L_{ba} nor $T(L_{ba})$ meets A.]

M. H. A. Newman (Manchester).

Pastidès, N. Sur les points fixes des transformations cycliques des domaines plans. Ann. Sci. École Norm. Sup. (3) 68, 169–184 (1951).

Let D be a simply connected domain in the plane whose boundary consists of more than one point, and let f be a periodic homeomorphism on D of period $n>1$. The author proves that if f preserves orientation, then its fixed point set consists of a single point, and that if f reverses orientation, then $n=2$ and the fixed point set is an open arc. No reference is made to any of the literature on periodic maps. For example, a theorem of Kerékjártó and Eilenberg [cf. Eilenberg, Fund. Math. 22, 28–41 (1934)] implies that a periodic map on the plane is equivalent to either a rotation or a reflection in a line. This, of course, implies the author's results.

E. E. Floyd (Charlottesville, Va.).

Boothby, William M. The topology of regular curve families with multiple saddle points. Amer. J. Math. 73, 405–438 (1951).

W. Kaplan [Duke Math. J. 7, 154–185 (1940); these Rev. 2, 322] has shown that any regular curve family filling the plane is the family of level curves for some continuous real function defined over the plane and without relative extrema. In the present paper this theorem is used as a lemma to establish the same theorem in case the curve family is regular except for singularities of the multiple saddle point type. The author announces that the results of this paper will be used in a later paper to extend, to curve families that are regular except for singularities of the multiple saddle point type, other results due to Kaplan [Trans. Amer. Math. Soc. 63, 514–522 (1948); Lectures in Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 299–301; these Rev. 9, 606; 3, 135] connecting regular curve families with harmonic functions and with the family of solutions of a system of differential equations.

W. R. Utz (Columbia, Mo.).

Whyburn, G. T. An open mapping approach to Hurwitz's theorem. Trans. Amer. Math. Soc. 71, 113–119 (1951).

Soit O l'ensemble des transformations continues, ouvertes et zéro-dimensionnelles (transformations intérieures, ou "light interior" au sens de l'auteur): $f(A)\subset B$ compactes dans A. Si A et B sont des variétés à deux dimensions, on sait que chaque f possède un degré $k(f)$ fini. Les théorèmes démontrés dans ce mémoire ont pour but d'étendre cette propriété individuelle des éléments de O à l'ensemble des transformations f; ce qui revient, dans un certain sense, à établir la continuité de la fonction $k(f)$ par rapport à f sur O. Ces théorèmes comportent une suite de corollaires intéressants qui conduisent, entre autres, à une démonstration et à une généralisation topologique d'un théorème classique de Hurwitz sur les fonctions analytiques, concernant les zéros des termes d'une suite uniformément convergente quand ces termes sont suffisamment rapprochés de leur fonction limite. Qu'il soit permis de remarquer que le lemme 1 de la page 116, assez riche en conséquences, a encore été démontré par le référent [Fund. Math. 13, 186–194 (1929), p. 194].

S. Stoïlow (Bucarest).

Keldyš, Lyudmila. Zero-dimensional mappings which increase dimension. Mat. Sbornik N.S. 28(70), 537–566 (1951). (Russian)

Detailed exposition of results reviewed earlier [Doklady Akad. Nauk SSSR (N.S.) 68, 989–992 (1949); these Rev. 11, 381].

L. Zippin (Brooklyn, N. Y.).

Boltyanskiĭ, V. On a property of two-dimensional compacta. Doklady Akad. Nauk SSSR (N.S.) 75, 605–608 (1950). (Russian)

For any finite closed covering $\Sigma=(F_1, \cdots, F_s)$ of a compactum Φ denote by n_i the number of indices $j\neq i$ for which $F_i\cap F_j\neq 0$. The density of the covering Σ is max $\{n_1, \cdots, n_s\}$. The density of the space Φ is the minimum of those integers n such that for every ϵ there exist finite closed ϵ-coverings of Φ with density n. The author proves that the density of any 2-dimensional compactum is $\cancel{\leq 6}$ (and hence that the density of a square is $=6$) and asserts that it is known that the density of a 1-dimensional compactum is 2 or 3.

R. H. Fox (Utrecht).

$\geqq 6$

Nagumo, Mitio. A theory of degree of mapping based on infinitesimal analysis. Amer. J. Math. **73**, 485–496 (1951).

The theory of the mapping degree for a continuous mapping f in finite-dimensional Euclidean spaces is usually based on the approximation of f by simplicial mappings. In the present paper simplicial mappings are avoided, their place being taken by continuously differentiable mappings. Thus a purely "analytical" treatment of the degree theory is attained. *E. H. Rothe* (Ann Arbor, Mich.).

Nagumo, Mitio. Degree of mapping in convex linear topological spaces. Amer. J. Math. 73, 497–511 (1951).

Leray and Schauder [Ann. Sci. École Norm. Sup. (3) **51**, 45–78 (1934)] extended the theory of the mapping degree to mappings in Banach space of the form $y=x+F(x)$ with completely continuous $F(x)$. Leray [C. R. Acad. Sci. Paris **200**, 1082–1084 (1935)] proved also the theorem of the invariance of the domain for such mappings. The present paper deals with the same topics, the difference from these papers of Leray and Schauder being: (i) the treatment is more detailed; (ii) the degree theory is developed in convex linear topological spaces, and the invariance theorem is proved for convex linear metric spaces. *E. H. Rothe.*

Krasnosel'skiĭ, M. A. On an elementary topological theorem. Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 160–164 (1951). (Russian)

On the unit sphere S_n in Euclidean $(n+1)$-space, suppose for each $x\epsilon S_n$ there is given a set of $n+1$ linearly independent unit vectors $A_1x, \cdots, A_{n+1}x$ such that, for each i, A_ix constitutes a continuous vector field. Call $\mathbf{A}=\{A_1, \cdots, A_{n+1}\}$ a moving coordinate system on S_n. Considering A_i as a mapping of S_n into itself, let $\gamma_{\mathbf{A}}$ be the common degree of the A_i's. Let Φ be a continuous field of non-zero vectors on S_n. If $\Phi(x)$ has coordinates $\psi_i(x)$, $1\leqq i\leqq n+1$, with respect to $\mathbf{A}$, let $\Psi(x)$ be the vector with the same coordinates with respect to the fixed coordinate axes. Then the theorem of the title is that $\gamma_\Phi=\gamma_{\mathbf{A}}\pm\gamma_\Psi$, where the γ's are the degrees of the corresponding vector fields, and where the sign depends upon whether $A_1x, \cdots, A_{n+1}x$ is positively or negatively oriented. It is pointed out that an $\mathbf{A}$ with $\gamma_{\mathbf{A}}\neq 0$ exists for $n=2, 4$. Among the applications made is a proof, for $n=2, 3$, of a well-known theorem of Borsuk [Fund. Math. **20**, 177–190 (1933)] showing the oddness of the degree of certain vector fields on S_n. *E. E. Floyd.*

Borsuk, Karol. Set theoretical approach to the disconnection theory of the Euclidean space. Fund. Math. **37**, 217–241 (1950).

A purely set-theoretic proof is given of the theorem that if A and B are homeomorphic closed subsets of E^n (= Euclidean n-space), then E^n-A and E^n-B have the same number of components. The proof uses a multiplication of homotopy classes of mappings of a closed set $A\subset E^n$ into the $(n-1)$-sphere defined previously by the author [C. R. Acad. Sci. Paris **202**, 1400–1403 (1936)]. Without using any combinatorial topology, it is shown that this multiplication turns these homotopy classes into an abelian group. In particular, if A is the $(n-1)$-sphere, this group is cyclic with at least two different elements. (Actually, this group is infinite cyclic, but the proof of this fact, which is not needed here, seems to demand combinatorial arguments.) Next it is shown that if $A\subset E^n$, then the number of generators of this group is one less than the number of components of E^n-A, which proves the theorem. *E. G. Begle* (New Haven, Conn.).

Kuratowski, Casimir. Remark on an invariance theorem. Fund. Math. 37, 251–252 (1950).

Borsuk has shown [see the preceding review] by a purely set-theoretic argument that for any closed set A in S^n (= n-dimensional sphere), the number of components of S^n-A is a topological invariant of A. Here it is shown, also by a purely set-theoretic argument, that this theorem still holds when the restriction that A be closed is removed. Even more, if S^n is replaced by a locally connected compactum X for which the theorem holds for closed subsets, then it also holds for arbitrary subsets. *E. G. Begle.*

Gordon, I. I. Classification of the mappings of closed surfaces into the projective plane. Doklady Akad. Nauk SSSR (N.S.) **78**, 625–627 (1951). (Russian)

The object of this note is a theorem characterizing the homotopy classes of the maps of each of the compact two-dimensional manifolds, orientable and non-orientable, into the projective plane. This is proved by the relatively simple methods of an earlier note [same Doklady **65**, 441–444 (1949); these Rev. **10**, 617]. The results are stated to be consequences, but only very indirectly, of the more general work of Hsiang-Lin Shih [Duke Math. J. **10**, 179–207 (1943); these Rev. **5**, 48] which uses heavier machinery.

L. Zippin (Brooklyn, N. Y.).

Whitehead, J. H. C. On the theory of obstructions. Ann. of Math. (2) **54**, 68–84 (1951).

The author [same Ann. (2) **52**, 51–110 (1950); these Rev. **12**, 43], by use of his secondary boundary operator and his theory of composite chain systems, has been able to define the necessary pairing of the coefficient groups involved in order to get cup products and squaring operations. Using these principles, squaring operations are defined in this paper that enable him to get the homotopy classification of maps of a finite polytope K, $\dim K=n+1$, $n\geqq 2$, into a finite polytope Y, $\pi_i(Y)=0$, $i<n$. The method used extends when Y is an arbitrary space, $\pi_i(Y)=0$, $i<n$, and generalizes Postnikov's theorem [Doklady Akad. Nauk SSSR (N.S.) **71**, 1027–1028 (1950); these Rev. **11**, 676] where it is required that $\pi_n(Y)$ be finitely generated.

J. Dugundji (Princeton, N. J.).

Chang, S. C. Some suspension theorems. Quart. J. Math., Oxford Ser. (2) **1**, 310–317 (1950).

Theorem 1: Let Y be the space obtained from a connected space X and an n-cell σ^n by the identification induced by a mapping ϕ of the boundary $\dot\sigma^n$ into X. Consider the induced homomorphisms h_r: $\pi_r(\dot\sigma^n)\rightarrow\pi_r(X)$ and the injection homomorphisms i_r: $\pi_r(X)\rightarrow\pi_r(Y)$. If $r<2n-2$ and X is aspherical in the dimensions $2, \cdots, r-n+1$ then the kernel of i_r is the smallest subgroup of $\pi_r(X)$ that contains the image of h_r and all the Whitehead products $[\alpha, \beta]$, $\alpha\epsilon$ image of h_u, $\beta\epsilon\pi_v(X)$, $u+v=r+1$.

Theorem 3: Let X' be a connected space such that $Y\cap X'$ is a point. Let j_r and j_r' denote the injections $\pi_r(Y)\rightarrow\pi_r(Y\cup X')$ and $\pi_r(X')\rightarrow\pi_r(Y\cup X')$. Let j_r'' denote the injection into $\pi_r(Y\cup X')$ of the intersection of the images of the retraction-induced homomorphisms k_r: $\pi_r(X\cup X')\rightarrow\pi_r(X)$ and k_r': $\pi_r(X\cup X')\rightarrow\pi_r(X')$. If $r<2n-1$ and $X\cup X'$ is aspherical in dimensions $1, \cdots, r-n+1$ then $\pi_r(Y\cup X')$ is the direct sum of the images of j_r, j_r' and j_r''.

Theorem 2: Let Y be the space obtained from a connected space X and an n-sphere $\dot\sigma^{n+1}$ by the identification induced by a mapping ϕ' of a hemisphere σ^n of $\dot\sigma^{n+1}$ into X. Let f_r denote the induced homomorphism of $\pi_r(\dot\sigma^{n+1})$ into $\pi_r(Y)$.

If $r<2n$ and X is aspherical in the dimensions $1, \cdots, r-n$ then $\pi_r(Y)$ is the direct sum of the image of i_r, the image of f_r, and the groups generated by the Whitehead products $[\alpha, \beta]$, $\alpha\varepsilon$ image of f_u, $\beta\varepsilon$ image of i_v, $u+v=r+1$.

R. H. Fox (Utrecht).

Burger, E. Über die Dualitätssätze für Homotopieketten. Ann. of Math. (2) 54, 56–67 (1951).

The object of this paper is to obtain analogs, in the Reidemeister homology theory, of the Eilenberg-MacLane duality relations between the ordinary homology and cohomology groups of a complex. As a preliminary, group extensions of groups admitting a ring of operators are studied. Equivalence classes of extensions of the Ω-module G by the Ω-module H are defined and form a group, Ext (G, H). The Eilenberg-MacLane formula,

$$\text{Ext}\,(G, H) \approx \text{Ophom}\,(R, G)/\text{Ophom}\,(R, G; F),$$

is shown to hold, where H is operator-isomorphic to the factor group of the free Ω-module F by the submodule R, Ophom (R, G) is the group of operator-homomorphisms of R into G, and Ophom $(R, G; F)$ is the subgroup of those which may be extended to operator-homomorphisms of F into G. If Ω is commutative, Ext (G, H) admits Ω as a ring of operators.

This preliminary theory is then used to obtain relationships between the Reidemeister homology and cohomology groups, Δ^r and ∇^r respectively, of a complex with "Überdeckung" Ω. Given certain restrictions on Ω, the author proves, by small modifications of the relevant Eilenberg-MacLane argument, that

$$(1) \qquad \nabla^p \approx \text{Ophom}\,(\Delta^p, \Omega) + \text{Ext}\,(\Omega, \Delta^{p-1}).$$

To obtain relations between the group invariants of the ∇^r and Δ^r, it is, of course, necessary to choose Ω so that the structure of Ω-modules is known. Taking Ω as the ring of algebraic integers of an algebraic number field, the author shows that (1) holds in this case and uses the known structure of Ω-modules to obtain an explicit expression for ∇^p in terms of the invariants of Δ^p and Δ^{p-1}. The paper closes with the remark that formula (1) is valid when Ω is any semi-simple algebra.

P. J. Hilton (Manchester).

***Eckmann, Beno. Espaces fibrés et homotopie.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 83–99. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

Le but de cette conférence est de donner un exposé sommaire des méthodes d'homotopie dans la théorie des espaces fibrés en vue de quelques applications importantes plus ou moins immédiates, anciennes et récentes, à des problèmes de champs de vecteurs, d'algèbra linéaire topologique, propriétés d'homotopie des espaces de groupe, etc. (Extract from paper.)

R. H. Fox (Utrecht).

Eilenberg, Samuel, and MacLane, Saunders. Cohomology theory of Abelian groups and homotopy theory. III. Proc. Nat. Acad. Sci. U. S. A. 37, 307–310 (1951).

Suite de 2 notes antérieures [mêmes Proc. 36, 443–447, 657–663 (1950); ces Rev. 12, 350, 520]. Dans cette note III, les auteurs démontrent la conjecture de la note II. D'une façon précise, soit π un groupe abélien; le complexe $K(\pi, n)$ de la note I est remplacé ici par $K'(\pi, n)$ obtenu en égalant à zéro certaines cellules "dégénérées"; $K'(\pi, n)$ est homotopiquement équivalent à $K(\pi, n)$. D'autre part, la "bar construction" de la note II, qui, à chaque algèbre différentielle graduée L, associe un complexe $\mathcal{B}(L)$, est modifiée par un changement de graduation, ce qui donne une algèbre différentielle graduée $\mathcal{B}^+(L)$. On définit un homomorphisme d'algèbres différentielles

$$\mathcal{B}^+(K'(\pi, n)) \to K'(\pi, n+1)$$

qui est une "chain equivalence". Il en résulte que les complexes (multiplicatifs) $A'(\pi, n)$ définis, par récurrence sur n, par

$$A'(\pi, 1) = K'(\pi, 1), \quad A'(\pi, n+1) = \mathcal{B}^+(A'(\pi, n))$$

sont homotopiquement équivalents aux complexes $K'(\pi, n)$. Ils sont aussi (après modification de la graduation) homotopiquement équivalents aux complexes $A^{n-1}(\pi)$ de la note II. Ceci prouve la conjecture annoncée, et fournit aussi une nouvelle preuve du théorème de suspension de la note I.

H. Cartan (Paris).

***Hopf, H. Sur une formule de la théorie des espaces fibrés.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 117–121. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

Let E be a fibre bundle for which the base space B is a simplicial polyhedron and the fiber V^n is a compact, orientable n-dimensional manifold for which the homotopy groups $\pi_p(V^n)$ are trivial for $1 \leqq p \leqq k-1$ $(k>1)$. Assume further that the local systems of groups in B defined by the homotopy groups $\pi_r(V^n)$ are simple. Under these conditions, the primary obstruction to defining a cross section of E is a single $(k+1)$-dimensional cohomology class with coefficients in $\pi_k(V^n)$. In case the primary obstruction vanishes, there exist secondary and higher obstructions to the definition of a cross section over all of B. These higher obstructions may consist of a whole set of cohomology classes rather than a single cohomology class. In this note, the author states without proof a formula which in some cases gives information about the higher obstructions. In addition to the obstructions to the extension of a cross section mentioned above, the formula involves the "cup" product of two other kinds of obstructions: (a) the well-known obstruction to deforming two different cross-sections so as to make them coincide, and (b), an obstruction to separating as much as possible two different cross-sections. In the special case in which the fibre V^n is a 2-sphere, and the primary obstruction (a 3-dimensional cohomology class) vanishes, the formula gives complete information about the structure of the secondary obstruction set. By means of the formula, the author defines some new invariants of the bundle structure. The author promises that a complete proof of the formula together with some applications will appear in a future publication.

W. S. Massey.

Young, Gail S., Jr. A generalization of the Rutt-Roberts theorem. Proc. Amer. Math. Soc. 2, 586–588 (1951).

Using Čech homology with compact supports and field coefficients the author proves the theorem: Let S be a normal space acyclic in dimension $i+1$, and let Z^i be a cycle on a compact subset K of S. Let the compact set M in $S-K$ be the union of a collection $G=\{C_\alpha\}$ of closed sets satisfying the following: (1) for every α, $Z^i \sim 0$ in $S-C_\alpha$; (2) there is a set C which is the intersection of each two elements of G and which links no $(i+1)$-cycle of S; (3) no closed set which is a union of elements of G links any $(i+1)$-cycle; and (4) any closed set which is the union of more than one element of G can be split into two closed proper subsets which are unions of elements of G. Then $Z^i \sim 0$ in $S-M$. This result

generalizes a proposition due to J. H. Roberts and N. E. Rutt as reformulated by R. L. Moore [Foundations of point set theory, Amer. Math. Soc. Colloquium Publ., vol. 13, New York, 1932]. *A. D. Wallace* (New Orleans, La.).

Graeub, W. Die semilinearen Abbildungen. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950**, 205–272 (1950).

A homeomorphism of one space upon another is called a semilinear mapping if there are simplicial decompositions of the given spaces such that each simplex of the decomposition of the first space is mapped affinely upon some simplex of the decomposition of the second. The first six paragraphs are devoted to a careful combinatorial proof of a famous theorem of Alexander [Proc. Nat. Acad. Sci. U. S. A. **10**, 6–8 (1924)] in the following slightly generalized form: In euclidean 3-space R^3 any polyhedral 2-sphere can be transformed into any other polyhedral 2-sphere by a semilinear mapping (of R^3 on itself). In the course of this a number of intuitively obvious theorems about semilinear mappings are proved rigorously. Samples: In euclidean n-space R^n any polygonal arc can be transformed into any other polygonal arc by a semilinear mapping (of R^n on itself); in R^n any simple closed polygon can be "rotated" by a semilinear mapping of R^n. In R^3 any polyhedral 2-cell can be transformed into any other one by a semilinear mapping of R^3, and any semilinear mapping of one polyhedral 2-cell in R^3 upon another one can be extended to a semilinear mapping of R^3 on itself.

In §7 the author proves that two simple closed polygons in R^3 are combinatorially isotopic [cf. Reidemeister, Knotentheorie, Springer, Berlin, 1932, Ch. 1, §3] if and only if one can be transformed into the other by an orientation-preserving semilinear mapping of R^3 on itself. §8 complements a recent paper of Seifert [Math. Z. **52**, 62–80 (1949); these Rev. **11**, 196]. Seifert showed that two "elements with self-penetration" are equivalent (by semilinear mapping of R^3 on itself) if their bounding "doubled knots" (Schlingknoten) are equivalent and that the two "carrier knots" (Diagonalknoten) are equivalent if the two "elements" are equivalent. The author shows that conversely if the two carrier knots are equivalent and if the "self-intersection" and "twisting" numbers of the "elements" are equal then the "elements" are equivalent (and hence, of course, the bounding "doubled knots" are equivalent). In §9, using the Tietze-Smith-Veblen-Alexander deformation theorem [Proc. Nat. Acad. Sci. U. S. A. **9**, 406–407 (1923)] the author shows that any homeomorphism of the plane on itself can be transformed by an ϵ-isotopy into a semilinear mapping. Thence he derives the result that any orientation-preserving homeomorphism of the plane upon itself is isotopic to the identity. *R. H. Fox* (Utrecht).

Graeub, Werner. Die semilinearen Abbildungen. Arch. Math. **2**, 382–384 (1950).

Summary of the preceding paper.

GEOMETRY

Goormaghtigh, R. Terminologie dans la géométrie du triangle et du tétraèdre. Mathesis **60**, 24–31, 116–123, 187–195, 263–274 (1951).

***van Veen, S. C. Passermeetkunde. De constructies van Mascheroni. [Geometry of Compasses. The Constructions of Mascheroni].** J. Noorduijn en Zoon, Gorinchem, 1951. 184 pp. 5.50 florins.

After an account of the history of geometrical constructions in general, the author draws attention to the practical advantage of the compasses over the ruler. His constructions, using the compasses alone, are well illustrated and neatly described. He employs the convenient symbol $A(BC)$ for the circle with center A and radius BC. In chapter I he shows how to construct the midpoint of a segment, for instance, and the sum of two given segments. Chapter II contains the basic constructions (e.g., the point of intersection of AB and CD) which enable him to prove the fundamental theorem of the geometry of compasses: Every Euclidean construction can be carried out with compasses alone. Chapter III emphasizes the fact that in many cases the number of circles involved can be greatly reduced by a fresh approach differing radically from the classical construction. In this spirit the author constructs the various centers of a triangle. In chapter IV and an appendix, a given circle is dissected into certain numbers of arcs, such as 5, 6, 10, 15 and 17. Chapter V is on inversion. Since all the straight lines in a classical construction can be simultaneously inverted into circles, this provides an alternative proof for the fundamental theorem of the geometry of compasses. Chapter VI contains approximate constructions for the duplication of a cube, the trisection of an angle, the quadrature and rectification of a circle, the regular heptagon and enneagon. For instance, an ingenious construction involving twelve circles will trisect any angle less than 30° with an error less than 1″.

Chapter VII deals with constructions on a sphere. Compasses are used to find the antipodes of a given point, the corresponding great circle, the great circle through two given points, the circle through three given points, and so on. It is easier than one might expect, to inscribe in the sphere each of the five Platonic solids. The above-mentioned appendix contains a simple proof that it is impossible (with compasses, or with ruler and compasses) to construct a regular enneagon; a fortiori, it is impossible to trisect an arbitrary angle. Another appendix (ten pages) is devoted to the life of L. Mascheroni (1750–1800), who proved that

$$\operatorname{li} x - \log(-\log x)$$

tends to Euler's constant when x approaches 1 from below, and who adorned his "La Geometria del Compasso" [Pavia, 1797] with a poetic dedication to Napoleon Bonaparte. It seems slightly unfair that the author spares only one short paragraph for G. Mohr who admittedly anticipated Mascheroni by more than a hundred years. Mohr's "Euclides Danicus" [Amsterdam, 1672; Copenhagen, 1928] might well have been added to the otherwise excellent bibliography of 27 works. The book ends with a recapitulation of the constructions (giving precise references to Mascheroni), a full table of contents, and a good index. *H. S. M. Coxeter* (Toronto, Ont.).

Mahler, K. On a question in elementary geometry. Simon Stevin **28**, 90–97 (1951).

The author proves without use of calculus the following result: Let γ_1 and γ_2 be two circular arcs through the points A, B. Let r_1 and r_2 be their radii. If $r_2 > r_1$ then γ_1 is longer than γ_2. The proof is long but elementary. It is possible to

give shorter proofs (a shorter proof was communicated to the reviewer by T. Aardenne-Ehrenfest). *P. Erdös.*

Droussent, Lucien. Sur une cubique circulaire circonscrite à un triangle. Bull. Soc. Roy. Sci. Liège **20**, 227–236 (1951).

Biernacki, Mieczyslaw. Sur le calcul des aires situées sur une sphère. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **4**, 19–21 (1950). (French. Polish summary)

Declaye, Gilberte. Sur les surfaces cubiques s'osculant le long d'une cubique gauche. Bull. Soc. Roy. Sci. Liège **20**, 107–113 (1951).

Vainstein, B. K. On vector models of crystal structures. Doklady Akad. Nauk SSSR (N.S.) **78**, 1137–1140 (1951). (Russian)

Palaj, Cyril. Sur la signification géométrique de certains invariants simultanés des coniques et des quadriques. Časopis Pěst. Mat. Fys. **75**, 159–177 (1950). (French. Czech summary)

Le sujet de cet exposé est la discussion et la recherche de la signification géométrique des invariants déduits des discriminants des formes quadratiques ternaires et quaternaires. L'auteur emploie systématiquement des déterminants cubiques et ainsi ressortit mieux les relations mutuelles de même que la signification géométrique.

H. A. Lauwerier (Amsterdam).

Weber, Werner. Der Begriff des Dreiecks bei apolaren Kurven zweiter Ordnung und Klasse. Collectanea Math. **3**, 121–135 (1950).

This is a careful analysis of the various degenerate cases of a triangle that is inscribed in one conic and self-polar for another. The author represents both the point conic $\sum\sum c_{ij}x_ix_j=0$ and the line conic $\sum\sum c_{ij}X_iX_j=0$ by the vector

$$(c_{00},\ \sqrt{2}\,c_{01},\ c_{11},\ \sqrt{2}\,c_{02},\ \sqrt{2}\,c_{12},\ c_{22})$$

in Euclidean 6-space, with the result that two perpendicular vectors represent a point conic and a line conic that are apolar. *H. S. M. Coxeter* (Toronto, Ont.).

Lombardo-Radice, Lucio. Sull'immersione di un piano grafico in uno spazio grafico a tre dimensioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 203–205 (1951).

This is a new proof of the theorem that if a graphical plane S_2 is Desarguesian, then it can be imbedded in a graphical 3-space S_3 [for the definitions cf. B. Segre, Lezioni di geometria moderna, vol. I, Zanichelli, Bologna, 1948; these Rev. **10**, 729]. Starting with S_2 the author actually constructs a space S_3 which contains a space S_2' isomorphic with S_2. The points of S_3 in certain homologies of S_2, and the lines and planes of S_3 are defined as classes of such homologies. The procedure extends to the imbedding of an S_{n-1} in an S_n for $n>3$. *D. B. Scott* (London).

Löbell, Frank. Der "Kern" als Basis komplexer Vektoren. Math. Z. **54**, 129–135 (1951).

In an earlier work [Math. Z. **52**, 759–769 (1950); these Rev. **12**, 123] the author exhibited an analogy between screw-displacements in hyperbolic space and vectors in complex Euclidean space. He now considers the hyperbolic analogues of a covariant basis and the reciprocal contravariant basis [G. Hessenberg, Math. Ann. **78**, 187–217 (1918)]. Each of these is a triad of screw-displacements whose axes have no common perpendicular. The six axes of the two triads are the sides of a rectangular skew hexagon in the hyperbolic space. *H. S. M. Coxeter.*

Rajčić, Lav. Étude sur des constructions fondamentales planimétriques de la géométrie de Lobatchevsky, par des méthodes synthétiques de la géométrie projective. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **5**, 57–120 (1950). (Serbo-Croatian. French summary)

Assuming the projective properties of a real conic and the corresponding constructions, the author introduces the group of hyperbolic displacements in the Lobatchevsky model by generating it by its "reflections." Congruence and metric are defined in the traditional way, and a detailed account is given of the elementary constructions.

F. A. Behrend (Melbourne).

Morduhaĭ-Boltovskoĭ, D. The theorem of Poncelet in the Lobačevskiĭ plane and elliptic integrals. Doklady Akad. Nauk SSSR (N.S.) **77**, 961–964 (1951). (Russian)

Jacobi gave the geometrical interpretation of Euler's well-known differential equations. Poncelet's theorem proves to be a simple corollary of this. The author shows that these considerations can be extended to the Lobachevsky geometry. In particular Poncelet's theorem remains valid in the L-plane. *H. A. Lauwerier* (Amsterdam).

Cassina, Ugo. Ancora sui fondamenti della geometria secondo Hilbert. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **12**(**81**), 71–94 (1948).

In a previous paper [Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **72**, 337–357, 358–372 (1937)] the author analyzed the sixth edition of Hilbert's Grundlagen der Geometrie [Teubner, Leipzig, 1923]. The present paper is concerned with the first two groups of axioms as formulated in the seventh edition [1930]. The axioms are rewritten in Peano's symbolic notation (of which however no further use is made) and broken up into simple constituents revealing some trivial redundancies. The author's opinion that the groups of axioms are incomplete because lines and planes must be construed as classes of points, appears to rest on a misunderstanding of Hilbert's intentions. The author then gives his own version of a system of independent axioms and proves the equivalence of his system with that of Hilbert.

F. A. Behrend (Melbourne).

Bing, R. H. An equilateral distance. Amer. Math. Monthly **58**, 380–383 (1951).

A metrization $D(p,q)$ of a topological space S is called equilateral provided there corresponds to each pair of points p, q of S a point r such that $D(p,q)=D(q,r)=D(p,r)$. The author shows (1) the euclidean metric is not equilateral for bounded subsets of the euclidean plane, and (2) any arc has an equilateral metrization. *L. M. Blumenthal.*

Convex Domains, Extremal Problems, Integral Geometry

Klee, V. L., Jr. On certain intersection properties of convex sets. Canadian J. Math. **3**, 272–275 (1951).

(I) Consider a collection of $n+1$ convex sets C_i, all lying in an m-dimensional real vector space V, with the property that every n of these sets have an interior point in common,

though not all $n+1$ of them do. It is shown that a linear variety of dimension $m-n$ (deficiency n) exists which avoids the interior of each C_i but which has, in any given direction away from itself, a translate meeting the interior of some C_i. Case $n=1$ of this result is the well-known separation theorem.

(II) Consider a collection of arbitrary cardinality of compact convex sets C_i in V. The following statements are shown to be equivalent: (1) every n of the C_i have a point in common; (2) every variety of deficiency n lies in some variety of deficiency $n-1$ which meets all the C_i; (3) every variety of deficiency $n-1$ has a translate which meets all the C_i; that (1) implies (2) is Horn's generalization of Helly's theorem [Bull. Amer. Math. Soc. 55, 923–929 (1949); these Rev. 11, 200]. (I) is used in proving that (3) implies (1).

W. Gustin (Bloomington, Ind.).

Hlawka, Edmund. Integrale auf konvexen Körpern. III. Monatsh. Math. 55, 105–137 (1951).

[For the first two parts see same Monatsh. 54, 1–36, 81–99 (1950); these Rev. 12, 197, 198.] In this paper the following integral inversion theorem is obtained. Let $f(x)$, where $x=(x_1, x_2, \cdots, x_m)$, be the distance-function and $H(u)$, where $u=(u_1, \cdots, u_m)$, the support-function (Stützfunktion) of the convex body K: $f(x)\leqq 1$ in m-dimensional space. It is supposed that $f(x)$, $H(u)$ are analytic except possibly at the origin, and that all tangent planes to K have contact of the first order only. Let $\Phi(t)$ be defined in $0<t<\infty$ such that $\Phi(t)t^{(m-1)/2}$ is absolutely Riemann integrable in $(0, \infty)$ and that $\int_0^\tau|\Phi(t)|t^{(m-1)/2}dt=o(\tau^k)$ for $\tau\to 0+$ and some $k>0$. Then

$$\mathfrak{J}(x)=\lim_{T\to\infty}\int_{f(y)\leqq T}\Phi(f(y))e^{ixy}dy$$

(where $xy=x_1y_1+\cdots+x_my_m$) exists for all x and

$$(2\pi)^{-m}\lim_{T\to\infty}\int_{H(u)\leqq T}\mathfrak{J}(u)e^{ixu}du=\tfrac{1}{2}\{\Phi(f(x)+)+\Phi(f(x)-)\}$$

provided that $\Phi(t)$ is of bounded variation in a neighbourhood of $t=f(x)$. As an application the proof of a theorem of C. G. Esseen on probability distributions is shown to generalize from spheres to bodies of the type K [Acta Math. 77, 1–125 (1945), chapter VII, theorem 1; these Rev. 7, 312].

J. W. S. Cassels (Cambridge, England).

Hadwiger, H. Zur Inhaltstheorie der Polyeder. Collectanea Math. 3, 137–158 (1950).

Elementary proof of the well-known theorem, that the content of n-dimensional polyhedra is the only polyhedron functional which is translation invariant, additive, nonnegative, and has the value 1 for a unit cube.

B. Jessen (Copenhagen).

Kneser, Martin. Über den Rand von Parallelkörpern. Math. Nachr. 5, 241–251 (1951).

In an n-dimensional Riemannian space [a space with $\sum g_{ik}dx_idx_k$ for the square of the element of length], let K be a set of points, let K_h denote, for $h>0$, the set of the points joinable to K by paths of length $\leqq h$, let T_h denote the frontier of K_h, and let $a(h)$ denote the $(n-1)$-dimensional Minkowski area of T_h. The author's main results, which improve on the known fact that T_h has n-dimensional measure zero, are as follows. (1) $a(h)$ is finite whenever T_h is compact; in particular this is so for all h if K is bounded and the space complete. (2) Given any compact subset M of the space, there exist constants c, C such that the relations $K\subset M$ and $0<h<c$ imply $ha(h)\leqq C$. (3) If the space is Euclidean, $ha(h)\to 0$ as $h\to 0$.

L. C. Young.

Habicht, Walter, und van der Waerden, B. L. Lagerung von Punkten auf der Kugel. Math. Ann. 123, 223–234 (1951).

The $N=12/(6-q)$ vertices of a regular polyhedron $\{3, q\}$ inscribed in the unit sphere are the centers of N small circles of angular radius θ, as closely packed as possible. The relations

$$\cos\theta=\tfrac{1}{2}\operatorname{cosec}\pi/q,\quad q=6(N-2)/N$$

hold for $q=2, 3, 4, 5$ ($N=3, 4, 6, 12$). For other values of N the packing cannot be so close, but we have instead

$$\cos\theta>\tfrac{1}{2}\operatorname{cosec}\frac{N}{N-2}\frac{\pi}{6}.$$

The authors replace this by a rigorous argument. They admit that the same result was obtained independently by L. Fejes Tóth [Jber. Deutsch. Math. Verein. 53, 66–68 (1943); these Rev. 8, 167]. They deduce that the surface Φ of the smallest sphere that will accommodate N nonoverlapping circles of straight diameter 1 satisfies

$$\Phi=\pi\operatorname{cosec}^2\theta\geqq\tfrac{1}{2}\sqrt{3}N+(5\pi/6-\sqrt{3})+O(N^{-1}).$$

By an ingenious use of gnomonic and stereographic projection, they find that $\Phi=\frac{1}{2}\sqrt{3}N+O(N^{\frac{1}{2}})$. Exact results for small values of N have been published recently [Schütte and van der Waerden, Math. Ann. 123, 96–124 (1951); these Rev. 13, 61].

H. S. M. Coxeter (Toronto, Ont.).

Rényi, A., Rényi, C., et Surányi, J. Sur l'indépendance des domaines simples dans l'espace Euclidien à n dimensions. Colloquium Math. 2, 130–135 (1951).

The sets $E_1, \cdots, E_n$ are said to be independent if no intersection $F_1, \cdots, F_n$ is empty, where each F_j is either E_j or its complement. The following theorems are proved. (A) For each $n\geqq 1$ the maximum number of open n-dimensional intervals (sides parallel to the coordinate axes) in Euclidean n-space $E^{(n)}$ is $2n$. (B) For each $n\geqq 1$ the maximum number of n-dimensional independent spheres in $E^{(n)}$ is $n+1$. (C) If $N(k)$ is the maximum number of polygonal open convex independent domains in the plane, each polygon having $\leqq k$ sides, then $\lim_{k\to\infty} N(k)/\log k=1/\log 2$.

J. L. Doob (Urbana, Ill.).

Stewart, B. M. The two-area covering problem. Amer. Math. Monthly 58, 394–403 (1951).

Let A_1 and A_2 be plane regions bounded by Jordan curves C_1 and C_2. Let A_2 be fixed while A_1 can move rigidly in the plane. The paper deals with the problem to determine the position of A_1 for which the area A of the union $A_1\cup A_2$ is minimum (or, what amounts to the same, for which the area of the intersection $A_1\cap A_2$ is maximum). Let x, y be rectangular coordinates of a point fixed in A_1, and let θ be the direction angle of a ray fixed in A_1. Then the position of A_1 depends on these 3 parameters. Assume that C_1 and C_2 can intersect in only a finite number of points. Then the partial derivatives A_x, A_y, A_θ of A exist and are continuous. These derivatives are expressed in terms of the intersection points P_i ($i=1, \cdots, 2n$) of C_1 and C_2 taken in their natural order on C_1. Thus the positions of A_1 for which A is stationary are characterized by certain properties of the polygon $P_1P_2\cdots P_{2n}$. The case of a circle A_1 and an oblique triangle A_2 is discussed in detail and relations to the geometry of the triangle are established.

W. Fenchel (Copenhagen).

Kuipers, L., and Meulenbeld, B. Two minimum problems. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 135–142, 143–151, 237–242 (1951).

L. Fuchs und Ref. [Compositio Math. 8, 61–67 (1950); diese Rev. 11, 455] haben bewiesen, dass die Summe und damit auch das einfache arithmetische Mittel der Inhalte des einem Kreise eingeschriebenen und des zugehörigen umgeschriebenen Polygons (die zwei Polygone berühren einander an der Kreisperipherie) ein absolutes Minimum für das Quadratpaar hat. Verf. untersuchen den bezüglichen arithmetischen Gewichtsmittelwert mit dem Gewicht p $(1>p>0)$ und finden, dass das Minimum (a) bei dem Dreieckpaar mit zwei gleichen Winkeln α und mit einem Winkel β ist, falls $1>p>p_3$; (b) bei dem regulären Dreieckspaar ist, falls $p_3>p\geqq\frac{2}{3}$; (c) bei dem Viereckspaar mit drei Zentralhalbwinkeln α und mit einem Zentralhalbwinkel β ist, falls $\frac{2}{3}>p>p_4$; (d) bei dem Quadratenpaar ist, falls $p_4>p\geqq\frac{1}{2}$; (e) bei dem Fünfeckspaar mit vier Zentralhalbwinkeln α und mit einem Zentralhalbwinkel β ist, falls $\frac{1}{2}>p>p_5$; (f) bei dem regulären k-eckspaar ist, falls $p_k>p>p_{k+1}$ $(k=5, 6, \cdots; \lim_{k\to\infty} p_k=\frac{1}{3})$; (g) nicht existiert, falls $\frac{1}{3}>p>0$ (in diesem Falle ist der Kreisinhalt die grösste untere Schranke). Dabei ist

$$p_k=\left[k\tan\frac{\pi}{k}-(k-1)\tan\frac{\pi}{k-1}\right]\Big/\left[\tfrac{1}{2}(k-1)\sin\frac{2\pi}{k-1}-(k-1)\tan\frac{\pi}{k-1}-\tfrac{1}{2}k\sin\frac{2\pi}{k}+k\tan\frac{\pi}{k}\right],$$

und α und β genügen den Gleichungen:

$$k\alpha_k+\beta_k=\pi,\quad \cos\alpha_k\cos\beta_k=[(1-p_{k+1})/2p_{k+1}]^{\frac{1}{2}},$$

$$\tfrac{1}{2}p_{k+1}(k\sin 2\alpha_k+\sin 2\beta_k)+(1-p_{k+1})(k\tan\alpha_k+\tan\beta_k)$$
$$=(k+1)\left[\tfrac{1}{2}p_{k+1}\sin\frac{2\pi}{k+1}+(1-p_{k+1})\tan\frac{\pi}{k+1}\right].$$

Der Beweis ist dem von L. Fuchs recht ähnlich obzwar naturgemäss mehr verwickelt und beruht auf den Ungleichungen von J. L. W. V. Jensen [Acta Math. 30, 175–193 (1906)] und von Hardy, Littlewood, und Pólya [Inequalities, Cambridge Univ. Press, 1934, S. 89–91] und auf einigen Eigenschaften der Funktion $f(x)=\frac{1}{2}p\sin 2x+(1-p)\tan x$. Ein ähnlicher Satz wird mit ähnlichem Beweis für das Gewichtsmittel der Umfänge des ein- und umgeschriebenen Polygons bewiesen, wo allerdings schon bei $p=\frac{1}{2}$ (ja sogar schon bei allem $p\leqq\frac{2}{3}$) kein Minimum sondern nur die Kreisperipherie als grösste untere Schranke existiert.

J. Aczél (Miskolc).

Komatu, Yûsaku. Isoperimetric inequalities. J. Math. Soc. Japan 2, 57–63 (1950).

The author gives function-theoretic proofs of the classical isoperimetric inequality and of an analogous inequality due to Bieberbach [Jber. Deutsch. Math.-Verein. 24, 247–250 (1915)] and he derives from his proofs slightly more precise results.

L. C. Young (Madison, Wis.).

Algebraic Geometry

Dedò, Modesto. Proprietà fondamentali delle quartiche piane dotate di punti doppi con tangenti inflessionali. Period. Mat. (4) 29, 11–32 (1951).

The author presents in this paper (which is intended mainly for high school teachers) some properties of the plane (irreducible) quartics, having two or three double points, with inflexional tangents. As the author points out in the 1st §, this is mainly an expository paper, and no claim of originality is made for the results presented. As an example of the topics dealt with, we quote the following theorem: Let C be a plane quartic, with two double points A, B (not cusps); if any three of the four tangents in A, B, are inflexional, the fourth has the same property. This result, and similar ones, are obtained in different ways, first using the harmonic homology, that transforms the curve C into itself, and then by means of the first properties of the geometry on a curve. A very elementary treatment of the groups of the projectivities of a plane curve into itself, and of the first theorems of the geometry on an algebraic curve completes the paper.

V. Dalla Volta (Rome).

Hutcherson, W. R. Point non parfait et courbes invariables. Bull. Soc. Roy. Sci. Liège 19, 485–489 (1950).

Hutcherson, W. R. A cyclic involution of period eleven. Canadian J. Math. 3, 155–158 (1951).

Arvesen, Ole Peder. On geometric addition of algebraic curves or surfaces. Norsk Mat. Tidsskr. 33, 54–59 (1951). (Norwegian)

A French version of this paper appeared earlier [Norske Vid. Selsk. Forh. 12, 115–118 (1940); 22, 163–166 (1950); these Rev. 2, 13; 11, 685].

Piazzolla-Beloch, Margherita. Curve algebriche piane d'ordine $2n$, con due punti multipli all'infinito di molteplicità n, (coniche generalizzate). Ann. Univ. Ferrara. Parte I. 6, 91–101 (1947).

In this paper the author studies curves of order $2n$ with two n-fold points at infinity. These curves, called n-conics, have many properties analogous to those of conics. For example, there are three types of n-conics, hyperbolic, elliptic, or parabolic, according as the two n-fold points at infinity are real and distinct, conjugate imaginary, or coincident. The elliptic and hyperbolic types have pairs of conjugate diameters belonging to an involution in which one pair, perpendicular to each other, are the principal diameters. In the parabolic type, the diameters are all parallel to each other.

T. R. Hollcroft (Aurora, N. Y.).

Rosina, Bellino Antonio. Sulle quartiche algebriche piane con due punti doppi all'infinito (biconiche). Ann. Univ. Ferrara. Part I. 7, 13–37 (1948).

The biconics (quartic curves with two double points at infinity) treated in this paper are a special case for $n=2$ of the n-conics defined and discussed by Margherita Piazzola-Beloch [see the paper reviewed above]. The properties of n-conics become those of biconics for $n=2$. The bicircular quartic is a special case of the elliptic biconic. Other special cases are discussed including rational biconics.

T. R. Hollcroft (Aurora, N. Y.).

Rosina, Bellino Antonio. Sopra alcuni tipi di curve algebriche piane per le quali ad un diametro generico corrispondono due direzioni distinte. Ann. Univ. Ferrara. Parte I. 6, 103–119 (1947).

The author obtains certain algebraic plane curves such that to any given diameter correspond two distinct conjugate diameters, by first solving the problem: Given a diameter whose conjugate has the slope m, to find the curves for which such a diameter corresponds to two conjugates of

slopes m and $-m$. Two cases occur according as the order of the curve is even or odd. Curves are also found which have one diameter of given direction conjugate to all directions in the plane, or, in other words, a unique diameter whose conjugate is indeterminate. *T. R. Hollcroft.*

Rosina, Bellino Antonio. Sopra un quesito proposto da Steiner. Ann. Univ. Ferrara. Parte I. 6, 137–144 (1947).

Jacob Steiner [J. Reine Angew. Math. 47, 106–108 (1854), p. 106 = Gesammelte Werke, vol. 2, Berlin, 1882, p. 599] proposed the following question: "How many diameters does a quartic curve possess (each of) which makes an angle α with its conjugate direction? How many for $\alpha=90°$? For $\alpha=0$ we have the four asymptotes." The author solves the problem for quartics and by generalization obtains the following results for curves of order n. An algebraic curve of order n has in general n diameters each of which makes with its conjugate a given angle α. Particular systems of curves of order n can be found for which every diameter makes the same angle α with its conjugate. Both of these statements hold for $\alpha=90°$. *T. R. Hollcroft* (Aurora, N. Y.).

Longhi, Ambrogio. Sulle sviluppabili osculatrici delle curve razionali iperspaziali. Comment. Math. Helv. 25, 131–139 (1951).

Dans une recherche précédente [même journal **24**, 196–203 (1950); ces Rev. **12**, 736], l'auteur a donné une formule générale qui permet de calculer le nombre des groupes possédant des éléments multiples quelconques, dans une série algébrique appartenant à une courbe algébrique rationnelle douée d'un certain nombre connu de points de rebroussement ordinaires. Cette formule est appliquée ici à une telle courbe (rationnelle, irréductible, $C_n{}^\rho$, d'ordre n, d'un espace S_r à r dimensions, possédant ρ points de rebroussement ordinaires en position générale, et en outre, des hyperplans hyperosculateurs ordinaires et des points multiples ordinaires à branches non singulières) pour calculer: A) la classe de la variété formée par les hyperplans de S_r ayant chacun avec $C_n{}^\rho$ un certain nombre donné de points de contact d'ordres donnés, ces points de contact étant tous différents des points de rebroussement de $C_n{}^\rho$; B) l'ordre de la variété lieu des points appartenant chacun à un certain nombre donné d'espaces osculateurs de $C_n{}^\rho$, de dimensions données. Le problème A) était déjà résolu, pour une courbe $C_n{}^\rho$ dépourvue de points de rebroussement, par une formule de De Jonquières; sa résolution générale était connue seulement dans l'espace ordinaire. La résolution du problème B), qui est le but principal de l'auteur n'était pas encore connue pour $r>3$, sauf dans le cas d'une courbe $C_4{}^\rho$ de S_4. Les formules obtenues par l'auteur sont assez compliquées et ne peuvent pas être répétées ici. De ces formules découlent une foule de conséquences particulières, se rapportant par exemple à la détermination de l'ordre des variétés multiples appartenant à la variété lieu des espaces, d'une dimension donnée, osculateurs à $C_n{}^\rho$. *E. Togliatti* (Gênes).

Gaeta, Federico. Nuove ricerche sulle curve sghembe algebriche di risiduale finito e sui gruppi di punti del piano. Ann. Mat. Pura Appl. (4) 31, 1–64 (1950).

This paper makes important advances in the theory of curves in space. The author first establishes, by means of the theory of syzygies of the corresponding H-ideal, the following theorems: (a) An arithmetically normal curve C in S_3 determines an integer ρ ($\geqq 0$) such that the ideal of polynomials vanishing on C is generated by the $(\rho+1)$-rowed minors of a matrix $\mathfrak{C}$ of $\rho+1$ rows and $\rho+2$ columns, the general element c_{ij} of which is a homogeneous polynomial of order $\mu_{ij}=\mu_{i1}+\mu_{1j}-\mu_{11}$, which may be taken in a normal form in which $\mu_{11}\leqq\mu_{12}\leqq\cdots\leqq\mu_{1,\rho+2}$, $\mu_{11}\geqq\mu_{21}\geqq\cdots\geqq\mu_{\rho+1,1}$. (b) If y_i is the minor got by suppressing the ith column of $\mathfrak{C}$, then the surfaces $y_1=0$, $y_2=0$ meet, residually to C, in an arithmetically normal curve whose corresponding matrix is obtained by supporting the first two columns of b, transposing, and inverting the order of rows and columns. Whence, (c) the curve C is of finite residual ρ. These ideas enable the author to extend the definition of curves of finite residual to singular and degenerate curves (even with multiple components). The integers μ_{ij}, μ_{i1} characterise a family of curves C_ρ of residual ρ, all of which present the same postulation for surfaces of every order, and every such family contains non-singular irreducible curves. The author obtains an expression for the dimension of a family of curves, and shows that, for every ρ, there are a finite number of regular families (for which the dimension is $4N$, N being the order of the curves). Every family of curves of order N and genus p contains degenerate curves consisting of N lines meeting in $N+p-1$ nodes.

Many of the results extend to V_{d-2} in S_d, and the author makes some applications to sets of points in a plane, for the details of which the paper should be consulted.

J. A. Todd (Cambridge, England).

Roth, Leonard. Sull'unirazionalità dell'intersezione di più quadriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 19–20 (1951).

L'intersection complète V^r de m quadriques de l'espace S^{m+r} est une variété unirationnelle. Dans le cas où $r\geqq\frac{1}{2}m^2-2$, elle peut se représenter par une involution d'ordre $2m-2$.

B. d'Orgeval (Grenoble).

Segre, Beniamino. Sull'esistenza, sia nel campo razionale che nel campo reale, di involuzioni piane non birazionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 94–97 (1951).

To avoid confusion, the author calls an algebraic variety birational if it is in birational correspondence with a linear space. The classical theorem of Lüroth states that over any commutative field every ∞^1 involution on a line is birational. Castelnuovo has proved the birationality of every plane ∞^2 involution over the field of complex numbers. The author shows, by constructing effective counter-examples, that the Castelnuovo theorem [Math. Ann. 44, 125–155 (1894)] does not hold over the field of real or rational numbers. *D. Pedoe* (London).

Godeaux, Lucien. Points de diramation des surfaces multiples. Bull. Soc. Math. Belgique 3 (1949–1950), 41–45 (1951).

Galafassi, Vittorio Emanuele. Superficie algebriche reali dotate di falde pari di prima specie. Rivista Mat. Univ. Parma 2, 115–121 (1951).

A real sheet of a real algebraic surface in three dimensions is defined to be even or odd according as it meets a general real line in an even or odd number of points, and an even sheet to be of the first or second kind according as it contains any odd circuits or not. It is shown that a surface whose general plane section is of genus p has at most $p+1$ even sheets of the first kind, and if it has this maximum number it is ruled; and conversely that for all p there exist ruled surfaces of genus p with $p+1$ even sheets of the first kind,

and with no real singularities. The maximum number δ_n of even sheets of the first kind for a nonsingular surface of order $2n$ is studied, and it is shown that for $n \geqq 2$

$$(n-1)^2+1 \leqq \delta_n \leqq (2n-1)(n-1)$$

and in particular $\delta_2=2$. *P. Du Val* (Athens, Ga.).

Baldassarri, Mario. Su una classe di superficie-modello di una trasformazione birazionale fra due piani. Ann. Mat. Pura Appl. (4) 31, 231–261 (1950).

The author shows that every non-ruled surface F_0^{2n+2} of order $2n+2$ in S_{n+4} $(n \geqq 1)$ with residual of genus zero (in the sense defined in his previous paper [Rend. Sem. Mat. Univ. Padova 20, 167–183 (1951); these Rev. 13, 62]) possesses one or more pairs of homaloidal nets of rational normal curves of order $n+1$, the two nets in a pair being residual with respect to the system of hyperplane sections. The S_{n+1}'s cutting out the curves of a pair of residual nets generate a cubic primal which contains a double V_n^6 with elliptic curve sections. The surface can thus be regarded as the image of corresponding point-pairs in a Cremona transformation T_n of degree n between two planes. Such surfaces can be generated as the locus of intersection of corresponding spaces in $n+1$ projective stars (∞^2) three of spaces S_{n+1} and the rest of hyperplanes. Conditions are determined in order that a surface Φ determined by such a projectivity, which in general is of order $2n+2+\binom{n-1}{2}$, should reduce to a surface of type F_0. These lead to a general type of analytical representation of a Cremona transformation T_n.

J. A. Todd (Cambridge, England).

Differential Geometry

✱König, R., und Weise, K. H. Mathematische Grundlagen der höheren Geodäsie und Kartographie. Erster Band. Das Erdsphäroid und seine konformen Abbildungen. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1951. xviii+522 pp. DM 46; bound, DM 49.60.

The field of higher geodesy and cartography is one specialized branch of geometry and analysis which has been the subject of many particular investigations. Generalizations developed from it have usually resulted in advances in broader fields of pure mathematics. In this book the authors endeavor an elaborate generalization but successfully restrict themselves to the selected field. Approximately one-half of the content is concerned with mathematical preliminaries and explicit formulation of characteristic relations for the generalized transformations discussed. The major contribution is the introduction of general complex vector coordinates for the spheroid such that a direct cartographic transformation can be stated and such that methods of complex function theory can be applied. As a treatment of cartography there is an unexpected but beneficial lack of reference to specific map projections. The analysis rests on the properties of three basic complex vector surface coordinates for the spheroid. These are the complex longitude, the complex latitude, and the complex meridian arc length. They give in direct projection respectively the Mercator projection, the transverse Mercator projection and the Gauss-Kruger projection. From these the entire pattern of conformal projections including various cylindrical, conical, and stereographic maps are shown to be obtained by selected complex function transformations. In each is given the explicit formulation of the projection both for the spheroid and the sphere as well as characteristic distortion parameters. A number of specialized problems such as projections from one spheroid to another are considered in detail.

The several chapter headings are: I. The spheroid. Calculation of curvatures for meridians and parallels, the surface area. II. The three fundamental surface variables for a surface of revolution, in particular for a spheroid and a sphere. III. The conformal mapping of two planes. IV. The geometrical relation between the three fundamental surface variables. V. Analytical representation (series) for the relation between the three fundamental surface variables and their exponential functions. VI. The conformal representation of the spheroid on a plane, sphere and spheroid. VII. The stereographic projection, conical projection, and the general arc projection of the spheroid. VIII. The transformation of orthogonal coordinate systems. IX. Various projection of the earth's spheroid on plane, sphere, and ellipsoid of revolution. X. Analytical aids. *N. A. Hall.*

Lips, L. A remark on certain twisted curves. Simon Stevin 28, 81–89 (1951).

The author studies two special cases of the system of twisted curves $\mathfrak{K}_0, \mathfrak{K}_1, \cdots$ ($\mathfrak{K}_n$ being the edge of regression of the polar surface of $\mathfrak{K}_{n-1}$). (A) The system for which $\tau_0 d\kappa_0/ds_0=C$; $\tau_1 d\kappa_1/ds_1=\lambda C$ and (B) the system for which $\tau_0 d\kappa_0/ds_0=\lambda\kappa_0$; $\tau_1 d\kappa_1/ds_1=\mu\kappa_1$, where κ_n, τ_n and s_n denote the curvature, torsion and arc of $\mathfrak{K}_n$ and λ, μ and C are constants. For both cases the intrinsic equations of the curves are given. The curves $\mathfrak{K}_n$ appear to be cylindrical helices.

J. Haantjes (Leiden).

Wunderlich, Walter. Über ein spezielles Dreiecksnetz aus Kegelschnitten. Monatsh. Math. 55, 164–169 (1951).

The author investigates the triangular nets which can be built up from conics. Many examples are already known of such nets which have been deduced from rectilinear nets by means of quadratic transformations. In this paper a further example is given of nets made up of conics being deducible from rectilinear nets by means of a cubic transformation. In the last paragraph the author points out that the cubic transformation used occurs in the study of reflection on an elliptic paraboloid, to which he has contributed [Monatsh. Math. 52, 13–37 (1948); these Rev. 9, 549].

E. T. Davies (Southampton).

Wunderlich, Walter. Beispiele für das Auftreten projektiver Böschungslinien auf Quadriken. Mat. Tidsskr. B. 1951, 9–26 (1951).

This paper continues the study of helices on surfaces of the second order already started by the author [Akad. Wiss. Wien, S.-B. IIa 155, 309–331 (1947); these Rev. 9, 612] and by F. Fabricius-Bjerre [Danske Vid. Selsk. Mat.-Fys. Medd. 25, no. 17, 21 pp. (1950); these Rev. 11, 741]. Whereas in the other papers the curves on surfaces of the second order whose tangents meet a certain conic c are considered and are called projective helices, in this paper the author considers any plane curve c, and proceeds to the determination of curves k on a regular quadric Φ whose tangents t meet the curve c.

There are two theorems, one concerning the case where c is any plane curve, and one for the case where c is a conic. The main result of the paper is that four classes of curves

which appear to have no relation to each other, can all be regarded as belonging to the class of projective helices.
E. T. Davies (Southampton).

Sakellariou, Nilos. Some observations on geodesic lines and curvatura integra. Bull. Soc. Math. Grèce 25, 120–129 (1951). (Greek. English summary)

Tomonaga, Yasuro. A generalization of Laguerre geometry. I. J. Math. Soc. Japan 2, 253–266 (1951).

The author attempts to generalize the Laguerre differential sphere geometry by making use of the tensor calculus. The tangential distance D between two hyperspheres and the condition that a hypersphere shall be tangent to a hyperplane are described in tensor language. Then a linear connection which leaves D invariant is introduced, and a curvature tensor is written out in the familiar way. "Frenet formulae" are derived in the study of one-parameter families of hyperspheres, and some theorems on special one-parameter families are presented. Then the author turns to $(n-1)$-parameter families, following the pattern expected in surface theory, and presents the concepts of conjugate directions, normal curvature, directions of curvature, and totally umbilical families. There are many typographical errors.
A. Schwartz (New York, N. Y.).

Pan, T. K. Hypergeodesics and dual hypergeodesics. Amer. J. Math. 73, 556–568 (1951).

L'auteur définit ce qu'il appelle "the extended relation R" et qui n'est autre que la polarité par rapport à la quadrique de Lie, la polarité fondamentale de l'école italienne [Fubini et Čech, Introduction à la géométrie projective différentielle des surfaces, Gauthier-Villars, Paris, 1931, ch. V]. Une hypergéodésique duale relative à une famille F d'hypergéodésiques est définie par la propriété que le point de rebroussement (ray point) en tout point P de cette courbe est sur la cubique transformée par la polarité fondamentale du cône enveloppe des plans osculateurs en P des courbes de F. L'auteur montre que ces hypergéodésiques duales constituent une famille à deux paramètres, duale de la famille F. La relation entre les deux famille est réciproque. Quelques propriétés des familles duales d'hypergéodésiques sont données, généralisant celles des systèmes axiaux (union curves). Les notations employées dans les calculs sont celles d'Eisenhart [An introduction to differential geometry, Princeton Univ. Press, 1940; ces Rev. 2, 154].
M. Decuyper (Lille).

Rollero, Aldo. Una proprietà delle calotte superficiali. Atti Accad. Ligure 6, 257–261 (1950).

Proof of the following theorem: a surface cap of order s (neighborhood of order s of a point on a surface) belongs always, for $s>2$, to algebraic surfaces of order $s-1$. The theorem does not hold for $s=2$. *E. Bompiani* (Rome).

Rollero, Aldo. Trasformazioni puntuali fra piani proiettivi e coppie di calotte superficiali del terzo ordine. Atti Accad. Ligure 6, 262–272 (1950).

A point-transformation ($\bar{x}=\bar{x}(x, y)$, $\bar{y}=\bar{y}(x, y)$) between two projective planes (x, y), $(\bar{x}, \bar{y})$ in the neighborhood of two corresponding regular points can always be represented by two surfaces in a space S_3 (putting $z=\bar{x}$, or $z=\bar{y}$). Of course this representation is not uniquely determined because of the arbitrary projective changes of coordinates in the given planes. Using this arbitrariness it is possible to reach the canonical forms already known. *E. Bompiani.*

Vaona, Guido. Ancora sul caso cremoniano delle trasformazioni puntuali. Boll. Un. Mat. Ital. (3) 6, 14–17 (1951).

The following theorem is derived: A necessary and sufficient condition that a point transformation between two projective spaces S_r, S'_r $(r>1)$ may be approximated up to the neighborhood of the 2nd order of a pair (O, O'), where the Jacobian vanishes, and its rank is k $(1\leqq k\leqq r-1)$ (O being of multiplicity $r-k$ for the Jacobian variety of the transformation), with a Cremona transformation is that the hypersurfaces corresponding to the hyperplanes through O' have in common a $(r-k)$-dimensional calotte of the 2nd order, whose center is O. *V. Dalla Volta* (Rome).

Čech, Eduard. Géométrie projective différentielle des correspondances entre deux espaces. II. Časopis Pěst. Mat. Fys. 75, 123–136 (1950). (French. Czech summary)

Čech, Eduard. Géométrie projective différentielle des correspondances entre deux espaces. III. Časopis Pěst. Mat. Fys. 75, 137–158 (1950). (French. Czech summary)

[Cf. part I of these papers [same Časopis 74, 32–48 (1950); these Rev. 12, 574].] In the papers II and III the correspondence is investigated for which there is a line a^i such that (1) $a^{[i} L^{j]}_{ke}=0$. Three different cases have to be considered. (a) $n=2$. Let K be a tangential projectivity with L^j_{ke}. Then for the most general tangential projectivity $^*K$ we have (2) $^*L^j_{ke}=L^j_{ke}+\delta^j_k\mu_e+\delta^j_e\mu_k$ and (3) $a^{[i}\,{}^*L^{j]}_{ke}=0$ yields $a^{[i} L^{j]}_{ke}=a^{[j}\delta^{i]}_k\mu_e+a^{[j}\delta^{i]}_e\mu_k$ and $a^k a^e a^{[i} L^{j]}_{ke}=0$ so that a^i is a characteristic line. Then μ_k can easily be found so that (3) holds. Hence, for $n=2$ the correspondence is always of the type mentioned above. (b) $n=3$. If K is a tangential projectivity for which (1) holds, then it is clear from (2) that for $n>2$ there is no tangential projectivity $^*K\neq K$ for which (3) holds. The lines a^i constitute a congruence. There are four different types of these congruences. (c) $n\geqq 4$. The correspondence for which (1) holds is obtained in the following way: Let S_n and S_n' be embedded in a S_{n+1} where we chose two arbitrary different fixed points P and $P'\neq P$. Project the generator point A of S_n from P in an arbitrarily chosen hypersurface (C) (in S_{n+1}) and project this projection from P' in S_n' and call this projection B. The required correspondence is $B\leftrightarrow A$. The author finds also another solution which depends on $2n$ arbitrary functions $(n\geqq 3)$.
V. Hlavatý (Bloomington, Ind.).

Ščerbakov, R. N. Projectively invariant repères of a line on a surface. Doklady Akad. Nauk SSSR (N.S.) 76, 805–808 (1951). (Russian)

The author constructs a projectively invariant repère for a point on a surface by requiring the ray MM_3 (M_3 not in the tangent plane) to be independent of the curve in the surface. He thus obtains a number of projective invariants whose vanishing yields the classical projectively invariant lines and curves associated with a point on a surface.
M. S. Knebelman (Pullman, Wash.).

Akivis, M. A. Pairs of T complexes. Mat. Sbornik N.S. 27(69), 351–378 (1950). (Russian)

A more detailed exposition of results previously published in a paper with the same title [Doklady Akad. Nauk SSSR (N.S.) 61, 181–184 (1948); these Rev. 10, 400].
M. S. Knebelman (Pullman, Wash.).

Blaschke, Wilhelm. Über Riemanngeometrie. Collectanea Math. 3, 73–104 (1950).

This paper employs orthogonal ennuples to give a unified expository account of the fundamental notions in Riemannian geometry. The treatment includes a discussion of Pfaffian forms, Grassmann algebra and matrices to the extent that those topics are useful in handling such topics as parallel displacement, geodesics, covariant derivatives and the curvature and Ricci tensors. *J. M. Thomas.*

Rauch, H. E. A contribution to differential geometry in the large. Ann. of Math. (2) **54**, 38–55 (1951).

The author proves that if the sectional curvatures of a complete Riemannian manifold R^n vary between positive bounds m, M such that $h<m/M\leqq 1$, where h is the positive root (approximately $\frac{3}{4}$) of the equation $\sin \pi\sqrt{h}=\frac{1}{2}\sqrt{h}$, then the universal covering Riemannian manifold of R^n is homeomorphic to the n-sphere. Previously it was known by results of the reviewer that the existence of a positive lower bound m for either the sectional curvatures or Ricci curvatures implies the compactness of R^n and its universal covering manifold. The proof of the author's theorem involves a study of the geodesic deviation along geodesics issuing from a point P of R^n. A second point $\bar{P}$ is found such that the n-cell formed by geodesics of fixed length issuing from $\bar{P}$ can be joined along an "equator" to the n-cell formed by geodesics issuing from P, so as to form a topological n-sphere. Complex projective space is an example to show that h cannot be replaced by a positive number less than $\frac{1}{4}$.

S. B. Myers (Ann Arbor, Mich.).

Rund, Hanno. Über die Parallelverschiebung in Finslerschen Räumen. Math. Z. **54**, 115–128 (1951).

As is well known, there are two points of view in which a Finsler space is considered as a point space or line element space whose tangent spaces are Minkowski spaces or Euclidean spaces, respectively. Many have developed the theory of Finsler spaces from the latter point of view, but the former was taken by very few besides Finsler himself. The author intends to develop from the first point of view the theory of parallel displacements in a Finsler space by geometrical considerations. He starts from the variation $[ik,j]_{(x,X)}X^iX^jdx^k$ of the local Minkowski measure $F^2(x,X)=g_{ij}(x,X)X^iX^j$ of a vector X^i with constant components between two consecutive points, where $g_{ij}(x,x')$ is the metric tensor derived from the fundamental function $F(x,x')$ and $[ik,j]_{(x,X)}$ denotes the Christoffel symbol of the first kind by means of $g_{ij}(x,X)$. Here g^{ij} is obtained from the relation $g^{ij}(x,y)g_{jk}(x,x')=\delta_k^i$, where $y_i=g_{ij}(x,x')x'^j$. The covariant differential of a vector field $X^i(x)$ is then defined as $DX^i-dX^i-d^*X^i$, where d^*X^i can be determined by the four conditions: (A) The scalar products of the vectors X^i and $X^i+d^*X^i$ with the tangent vectors at the points x^i and x^i+dx^i of the geodesic line passing through these two points, i.e.

$$g_{ij}(x,x')x'^iX^j=g_{ij}(x+dx,x'+dx')(x'^i+dx'^i)(X^j+d^*X^j);$$

(B) $dX^i-d^*X^i=0$, when $X^i=x'^i$ (i.e. any geodesic line is auto-parallel); (C) d^*X^i is linear in the X^h; (D) $Dg_{ij}(x,x')=0$. The geometrical consideration then leads us to

$$d^*X^i=-P^i_{hk}X^hdx^k,$$

where

$$P^i_{hk}=\left\{ {i \atop hk} \right\}_{(x,x')}-\tfrac{1}{2}g^{il}(x,y)\frac{\partial g_{hk}(x,x')}{\partial x'^l}\left\{ {l \atop jk} \right\}_{(x,x')}x'^j$$

and $\{{i \atop hk}\}_{(x,x')}$ denotes the Christoffel symbol of the second kind by means of $g_{ij}(x,x')$. The parallel displacement $DX^i=0$ from a point A to another point B (always along the geodesic line passing through these points) means that the parallel vector in the tangent Minkowski space $T_n(B)$ at B to a vector X^i in $T_n(A)$ at A appears as identical to X^i, when it is observed from A. But it should be noticed that the length of a vector X^i is not unchanged under a parallel displacement and that $P^i_{hk}(x,x')$ is not symmetric in two subscripts. At the end, the curvature of a curve in this space is introduced as an example for applications of the covariant differential and its geometrical meanings are stated. [Reviewer's remark. Although the ideas and methods in this paper are interesting and may be a contribution to the theory of Finsler spaces, the introduction of the last one of the four conditions for d^*X^i seems to the reviewer to be incomplete, because the covariant differential $Dg_{ij}(x,x')$ has never been defined, although the author defines that of a tensor $a_{ij}(x)$ whose components depend only on the position x^i but not on the direction x'^i.] *A. Kawaguchi.*

Vasil'ev, A. M. General invariant methods in differential geometry. Doklady Akad. Nauk SSSR (N.S.) **79**, 5–7 (1951). (Russian)

The problem of constructing differential-geometric objects as defined by Veblen-Whitehead has been studied by Vagner [same Doklady (N.S.) **69**, 293–296 (1949); these Rev. **11**, 461] and Laptev [ibid. **73**, 17–20 (1950); these Rev. **12**, 443]. The present paper outlines a general method of attack which essentially amounts to the study of solutions of completely integrable pfaffian systems, invariant under the fundamental group of the space. *M. S. Knebelman.*

✓ ***Ehresmann, Charles. Les connexions infinitésimales dans un espace fibré différentiable.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 29–55. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

This lecture gives an introduction to the theory of infinitesimal connections from a modern and general point of view. It's basis is the remark that, e.g., in a Riemannian manifold the vectors, on which the connection operates by parallel displacement, form a fibre bundle over the manifold. It begins with a discussion of the basic concepts concerning fibre bundles (covering homotopy, principal bundle, etc), with emphasis on differentiable bundles. Using the well known classical cases as motivation, an infinitesimal connection in a differentiable fibre bundle $E(B,F)$ with base space B and fibre F is then defined as a field C of n-dimensional ($n=\dim B$) contact elements (i.e. subspaces of the tangent spaces of the various points of E), transversal to the fibre, and satisfying the "covering" condition: If a is a path in B, with initial point x (and terminal point x'), then there exists an integral curve of C, covering a, and starting at an arbitrary point in the fibre F_x over x. The path a defines then a homeomorphism between F_x and $F_{x'}$, the "parallel displacement along a"; for $x=x'$ one obtains so the holonomy group. If the connection is integrable (i.e., if C is completely integrable), these homeomorphisms depend only on the homotopy classes of the paths. If the structure group G is a Lie group, the homeomorphisms are assumed to be isomorphisms of the G-structure on the fibre. This amounts to defining a connection as a connection $\bar{C}$ in the principal bundle, which is invariant under the operations of G on the principal bundle. A special type are the Cartan connections, where F is a homogeneous space (Klein space),

where dim $B = \dim F$, where B lies, as a cross section, in the bundle, and where in a certain sense each fibre F_x intangent to B at x [cf. the case of the tangent bundle to a manifold, where this is so by the very definition of the fibre F_x = tangent space at x], such a bundle is called "welded" to B. For the intrinsic ("covariant") derivative of a curve b in E one pulls b back into the fibre at its initial point by parallel displacement along the projection of b, and differentiates in the fibre. [Equivalently, one can decompose any tangent vector to E into the intrinsic component tangent to the fibre and the entrainment component in the element of the field C.] At the same time this defines development of b into the initial fibre. With a Cartan connection even curves in B can be so developed; this leads to the differential invariants of curves in B, namely, the invariants of the developed curves in F. A space with a Cartan connection is complete, essentially if every curve in F is development of a curve in B. The existence of Cartan connections with five B, F, G is discussed. Since it amounts to the existence of cross section in certain related bundles, one is led to obstruction cocycles. A number of examples, including the well known affine, Euclidean, projective and conformal connections, and many other applications and related topics are discussed.

H. Samelson (Ann Arbor, Mich.).

Milgram, A. N., and Rosenbloom, P. C. Harmonic forms and heat conduction. I. Closed Riemannian manifolds. Proc. Nat. Acad. Sci. U. S. A. 37, 180–184 (1951).

The result presented is as follows. If a differential form α on a compact domain satisfies the heat equation $\Delta\alpha - \partial\alpha/\partial t = 0$ in $0 \leqq t < \infty$ and if it is closed for $t=0$, it is also for $t>0$. Its periods remain constant, and for $t \to \infty$ converges to a harmonic form H_α having the same periods. But since a solution of the heat equations can be obtained for any boundary values at $t=0$, a new proof of Hodge's theorem ensues. In proving the last fact the authors claim to be able to replace a Fredholm integral equation by a simpler Volterra integral equation, but it is not immediately clear whether such a simplification is claimed for tensors only, or also for scalars as well, as an improvement over what has been known.

S. Bochner (Princeton, N. J.).

Bol, Gerrit. Alternierende Formen und halbinvariante Differentiation. Math. Z. 54, 141–159 (1951).

(I) Let w be a half invariant quantity of weight c, (1) $w^* = \rho^c w$ and let π_λ be a vector given up to transformation (2) $\pi_\lambda^* = \pi_\lambda - \partial_\lambda \log s$. Then (3) $\underline{d}w = dw + c\pi_\lambda w dx^\lambda$ is obviously half-invariant of the same weight c. Suppose that in every tangential plane of a surface $\mathbf{x}(x^1, x^2)$ in a projective three space there is given a straight line L not incident with the point $\mathbf{x}$, which itself is half invariant of weight 1. The requirement that (4) $\underline{d}\mathbf{x} = d\mathbf{x} + \pi_\lambda \mathbf{x} dx^\lambda$ be on L yields $\pi = \pi_\lambda dx^\lambda$ and π_λ transforms according to (2).

(II) Let $\mathbf{x}_i(x^1, x^2)$ $(i = 1, 2, 3, 4)$ be the vertex (half invariant of weight c_i) of a reference tetrahedron connected with $\mathbf{x}(x^1, x^2)$. Denote by Y_i the opposite face of the tetrahedron, and by T_i the tangential plane of the surface $\mathbf{x}_i(x^1, x^2)$ in $\mathbf{x}_i$ and put $L_i = Y_i T_i$. The line L_i yields the corresponding $\pi_i = \pi_{i\lambda} dx^\lambda$ as described above and consequently the half invariant point (of weight c_i) $\underline{d}\mathbf{x}_i = d\mathbf{x}_i + c_i \pi_i \mathbf{x}_i$ is incident with Y_i. Hence in the fundamental equations

$$(5) \qquad \underline{d}\mathbf{x}_i = \sum_{k=1}^{4} w_i^k \mathbf{x}_k \qquad (w_i^k = w^k{}_{i\lambda} dx^\lambda)$$

we have $w_1^1 = w_2^2 = w_3^3 = w_4^4 = 0$. Put

$$\underset{1}{\pi_i} = \underset{1}{\pi_{i\lambda}} dx^\lambda, \quad \underset{1}{w_i^k} = \underset{1}{w^k{}_{i\lambda}} dx^\lambda,$$
$$\underset{2}{\pi_i} = \underset{2}{\pi_{i\lambda}} dx^\lambda, \quad \underset{2}{w_i^k} = \underset{2}{w^k{}_{i\lambda}} dx^\lambda.$$

Then the integrability conditions of (5) namely

$$\underset{[2\ 1]}{d\pi_i}\mathbf{x}_i = \sum_{k,l=1}^{4} \underset{[1}{w_i^k} \underset{2]}{w_k^l} \mathbf{x}_l + \sum_{k=1}^{4} \mathbf{x}_k \underset{[2\ 1]}{dw_i^k}$$

may be split in

$$\underset{[2\ 1]}{d\pi_i} = \sum_{k=1}^{4} \underset{[1}{w_i^k} \underset{2]}{w_k^i},$$
$$(6) \qquad\qquad (i \neq l,\ k \neq i, l)$$
$$\underset{[2\ 1]}{dw_i^l} = \sum_{k=1}^{4} \underset{[2}{w_i^k} \underset{1]}{w_k^l}.$$

(III) The equations (5) and (7) may be thought of as fundamental equations of the theory of surfaces. The author shows in detail the efficiency of (5) and (7) especially if a conjugate net is given or if one uses asymptotic parameters.

V. Hlavatý (Bloomington, Ind.).

Patterson, E. M. On symmetric recurrent tensors of the second order. Quart. J. Math., Oxford Ser. (2) 2, 151–158 (1951).

A second order covariant tensor T_{ij} $(i, j = 1, 2, \cdots, n)$ in a Riemannian V_n is called a recurrent one if its covariant derivative $T_{ij,p}$ satisfies $T_{ij,p} = \kappa_p T_{ij}$ for some vector field κ_p. A V_n admitting a second order symmetric recurrent tensor other than a scalar multiple of the metric tensor g_{ij} will be denoted by $\mathcal{T}_n$. The author considers p independent vector fields $\lambda_\alpha{}^i$ $(\alpha = 1, 2, \cdots, p)$ defined over any V_n which satisfy the relations $\lambda^i{}_{\alpha,j} = A^\beta{}_{\alpha j} \lambda_\beta{}^i$. Those parallel fields of p-dimensional vector-spaces are called parallel p-planes over the V_n. In the case of indefinite metric some of the mutually orthogonal basis vectors $\lambda_\alpha{}^i$ $(\alpha = 1, 2, \cdots, p)$ of a parallel p-plane Π can be "null vectors" (isotropic vectors). If r of these vectors, say $\lambda_\rho{}^i$ $(\rho = 1, 2, \cdots, r)$ are isotropic and $p-r$ non-isotropic, then the vectors $\lambda_\rho{}^i$ form the "null-part" Π^* of Π. Another parallel plane determined by Π is the conjugate $(n-p)$-plane Π'. The author supposes now that $V_n = \mathcal{T}_n$ and T_{ij} is of the rank p (i.e. the matrix $\|T_{ij}\|$ is of rank p). Then $n-p$ linearly independent solutions λ^i of the equations $T_{ij}\lambda^j = 0$ do exist, which form the parallel $(n-p)$-plane associated with T_{ij}. The conjugate of the associated parallel plane is called the parallel plane generated by T_{ij}. In the case of non-isotropic and mutually orthogonal solutions λ^i the space admits a non-isotropic parallel plane. It is then reducible; i.e. there is a coordinate system in terms of which the metric is

$$ds^2 = g_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta + g_{\lambda\mu}(x^\nu) dx^\lambda dx^\mu;$$
$$\alpha, \beta, \gamma = 1, 2, \cdots, p;\ \lambda, \mu, \nu = p+1, \cdots, n.$$

In a space $\mathcal{T}_n$, any tensor $T^{(\nu)}_{ij}$ of the sequence defined by

$$T^{(\nu+1)}_{ij} = T^{\nu}{}_{ik} T_j{}^k, \quad T^{(1)}_{ij} = T_{ij} \quad (\nu = 1, 2, \cdots, n)$$

is a second-order symmetric recurrent tensor, satisfying

$$T^{(\nu)}_{ij,p} = \nu\kappa_p T^{(\nu)}_{ij}.$$

The process of forming the sequence is equivalent to taking powers of the matrix $(T_j{}^i)$. Each tensor $T^{(\nu)}_{ij}$, being recurrent, generates a parallel-plane Π_ν. Sums and intersections of the parallel planes Π_ν and their conjugates give rise to further sets of parallel planes. The characters (dimensionality and nullity-number of isotropic vectors in a normal basis) of all

these parallel planes are completely determined by the elementary divisors of the pencil $T_{ij}-\rho g_{ij}$ at any point of the space. *M. Pinl* (Dacca).

Littlewood, D. E. Differentiation in space-time. Proc. London Math. Soc. (2) 53, 43–56 (1951).

All possible invariantive differential operators, of the first and second orders for space of three or of four dimensions, are discussed, where the metric tensor is g_{ij}, subject to fully orthogonal or to rotational transformations. The familiar "grad", "div" and "curl" are shewn to be such necessary and sufficient operations of the first order in [3]-space; and their explicit generalizations for the metric g_{ij} are obtained. In [4], for Lorenz space-time there are five such operations in the fully orthogonal group which reduce to four in the rotation group. Second order differentiation is discussed, together with the matrix factorization of ∇^2 as instanced by the γ_i matrices of Dirac and the β-matrices of Kemmer (used in the theory of the meson and of electromagnetic radiation). A general tensor $A'_{i_1\cdots i_n}$ is symmetrized by taking the average of the $n!$ terms obtained by permuting the suffixes in all ways. The result, called $P_{i_1\cdots i_n}A'_{i_1\cdots i_n}$, is say $A_{i_1\cdots i_n}$. The components are functions of three variables x_1, x_2, x_3 and the suffixes take the values 1, 2, 3. For this symmetric A the operations are defined thus:

$$\operatorname{div} A=(\partial/\partial x_{i_n})A_{i_1\cdots i_n},$$
$$\operatorname{curl} A=P_{i_1\cdots i_{n-1}k}E_{i_njk}(\partial/\partial x_j)A_{i_1\cdots i_n},$$
$$\operatorname{grad} A=P_{i_1\cdots i_nj}\left(\frac{\partial}{\partial x_j}A_{i_1\cdots i_n}-\frac{n}{2n+1}g_{i_nj}\frac{\partial}{\partial x_k}A_{i_1\cdots i_{n-1}k}\right).$$

Here E_{i_njk} is the alternating tensor whose components take the values 0, ±1 as usual. The same holds for a spinor function of position except that the curl is slightly modified.

Proofs depend upon the algebraic formulae

$$\{n\}=[n]+\{n-2\},\qquad [n][1]=[n+1]+[n]+[n-1], \tag{1}$$

where $\{n\}$ denotes the trace of the matrix of transformation of the coefficients of a ternary n-ic, that is, here, of a function f of position that transforms in orthogonal 3-space like the nth power of a vector: and where $[n]$ is the corresponding trace of a simple function of position, namely one obtained from $f-\lambda g_{ij}\phi$ (where ϕ is the contraction of f with the metric tensor g_{ij}) by choosing the numerical λ so that a further such contraction of the whole expression is zero. For $n=1$, $\{1\}=[1]$; and this occurs in taking the trace of the transformation matrix for the vector $\partial/\partial x_i$. The product $[n][1]$ then implies, by its resolution into three simple terms in (1), that any first order differential expression is analysable into three or less terms called grad, curl, and div. The various possible second order derivatives are identified with the nine possible combinations grad grad, grad curl, $\cdots$, div div acting upon A. Each of these operations grad, curl and div, denoted indifferently by ∇ can be expressed as $(\partial/\partial x_i)T_i$ where T_1, T_2, T_3 are matrices each of p columns and q rows that operate on a function ϕ (of type $[\lambda]$ and with p components) in order to give the components of the function $\nabla\phi$. Various cases are considered.

With four variables similar treatment is available: and in the rotation group every rotation breaks up into a right-handed and a lefthanded rotation which are commutative operations. The algebraic identities

$$[p,q][1]=[p+1,q]+[p,q+1]+[p,q-1]+[p-1,q]$$

and

$$[p,q][2]=[p+2,q]+\cdots+[p-2,q]\ \text{(with nine terms)}$$

lead to four first and nine second order derivatives respectively, called $\operatorname{grad}_1 A$, $\operatorname{grad}_2 A$, $\operatorname{div}_2 A$, $\operatorname{div}_1 A$ and certain combinations made by pairing or repeating these first order ones. Here $[p,q]$ is the trace (or character) answering to a form of orders p and q in two sets of cogredient variables, so that $[p,0]$ denotes the previous $[p]$. Also p, q are integers such that $p\geqq 0$, $|q|\leqq p$; and in the case of spinors each of p and q is half an odd integer. The exceptional cases are worked out, as when $p=q=0$ for a scalar function of position in which only $\operatorname{grad}_1 A$ among the four possible first order derivatives exists. The factorization of ∇^2 is discussed, along with the appropriate matrices; and finally the case of the fully orthogonal group is sketched. *H. W. Turnbull.*

NUMERICAL AND GRAPHICAL METHODS

Uhler, Horace S. Approximations exceeding 1300 decimals for $\sqrt{3}$, $1/\sqrt{3}$, $\sin(\pi/3)$ and distribution of digits in them. Proc. Nat. Acad. Sci. U. S. A. 37, 443–447 (1951).

√ ***Salzer, Herbert E. Tables of $n!$ and $\Gamma(n+\frac{1}{2})$ for the first thousand values of n.** National Bureau of Standards Appl. Math. Ser., no. 16. United States Government Printing Office, Washington, D. C., 1951. vi+10 pp. $.15.

Tables of $n!$ and $\Gamma(n+\frac{1}{2})$ for $n=0(1)1000$ to 16 and 8 significant figures, respectively.

Bogert, B. P. Some roots of an equation involving Bessel functions. J. Math. Physics 30, 102–105 (1951).

Tables are given for the smallest positive $y=y(k)$ which satisfies $J_0(y)N_1(z)-N_0(y)J_1(z)=0$, where $z=ky$, and for the smallest positive $z(k)=ky(k)$ satisfying this when $y=k^{-1}z$. The values of k used include $k=1(.01)2(1)10, 20$, and the roots are given to 4 or 5D. Pivotal values of y were obtained by inverse interpolation using the British Association for the Advancement of Science Math. Tables, vol. VI, Bessel Functions, Pt. 1 [Cambridge Univ. Press, 1937] and these were subtabulated using the Everett formula. *J. Todd.*

Ricci, Lelia. Tavola di radici di basso modulo di un'equazione interessante la scienza delle costruzioni. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo, no. 296, 8 pp. (1951).

This table gives the first 11 non-zero roots z of $\sin z=\pm kz$ lying in the first quadrant of the z plane for $k=.1(.01)1$ to four decimals. Most of these roots are complex; the real ones are given separately. Asymptotic formulas are given for the real and imaginary parts of the nth root of the equation. *D. H. Lehmer* (Berkeley, Calif.).

Zavrotsky, A. Tables for solution of equations of the fifth degree. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. 13, no. 41, 51–93 (1950). (Spanish)

By a linear transformation every quintic equation can be thrown into the form (1) $x^5=px^3+qx^2+rx+1$. These tables give the least positive root or greatest positive root of (1) according as $p+q+r$ is negative or positive. These roots are

given to 5 decimals for all integer values of p, q, and r, not exceeding 10 in absolute value. *D. H. Lehmer.*

Fletcher, Alan. Tables of two integrals and of Spielrein's inductance function. Quart. J. Mech. Appl. Math. 4, 223–235 (1951).

In connection with the self-inductance of thin disk coils the following integrals arise

$$I=\int_\alpha^1 (K-E)dk, \quad J=\int_\alpha^1 (K-E)k^{-3}dk,$$

$$f(\alpha)=\frac{16\pi}{3(1-\alpha)^2}(I-\alpha^3 J),$$

K and E being the complete elliptic integrals of the first and second kind with modulus k. Tables are given for I, J and $f(\alpha)$ for $\alpha=0(.01)1$. *F. Oberhettinger.*

Wood, Harley. Tables for nearly parabolic elliptic motion. J. Proc. Roy. Soc. New South Wales **84** (1950), 134–150 (1951).

Tables are given for

$$6(\arcsin x-x)/x^3, \quad 6(1-(1-x^2)^{1/2})/x^2(1-x^2)^{1/2},$$
$$2(1-(1-x^2)^{1/2})/x^2$$

and for certain truncated series, all for the range 0(.001).6 on the argument. These functions occur in formulae derived by the author for facilitating ephemeris and perihelion time computations when the eccentricity is less than but close to 1 [Wood, same J. Proc. 83, 150–163 (1950); these Rev. 12, 365]. *R. G. Langebartel* (Urbana, Ill.).

Wood, Harley. Tables for hyperbolic motion. J. Proc. Roy. Soc. New South Wales **84** (1950), 151–164 (1951).

The author tabulates the functions counterpart to those tabulated in the paper of the preceding review for the case of the eccentricity greater than 1. *R. G. Langebartel.*

***Monte Carlo method.** National Bureau of Standards Appl. Math. Ser., no. 12. United States Government Printing Office, Washington, D. C., 1951. vii+42 pp. $.30.

Papers and discussion from a symposium on the Monte Carlo method, June 29–July 1, 1949, University of California, Los Angeles. The papers are devoted chiefly to uses of the method.

***Hartree, Douglas R., Lefschetz, S., Friedman, Bernard, and Dantzig, George B. Problems for the numerical analysis of the future.** National Bureau of Standards Appl. Math. Ser., no. 15. United States Government Printing Office, Washington, D. C., 1951. iv+21 pp. $.20.

A collection of four papers presented at the Symposia on Modern Calculating Machinery and Numerical Methods, July 29–31, 1948, University of California, Los Angeles: (1) Some unsolved problems in numerical analysis, by D. R. Hartree; (2) Numerical calculations in nonlinear mechanics, by S. Lefschetz; (3) Wave propagation in hydrodynamics and electrodynamics, by B. Friedman; (4) Linear programming, by G. B. Dantzig.

***Wilkes, Maurice V., Wheeler, David J., and Gill, Stanley. The Preparation of Programs for an Electronic Digital Computer.** Addison-Wesley Press, Inc., Cambridge, Mass., 1951. x+170 pp. (2 plates). $5.00.

The appearance of this book will mean very little to mathematicians in general, but it will mean a great deal to the few who are engaged in coding problems for large scale digital computers. Basically the book is a description of the current library of subroutines used on the EDSAC (Electronic Delay Storage Automatic Computer), at Cambridge, England. The choices they have made, their methods of organization of subroutines, and their emphasis on checking should prove valuable to others faced with the problem of operating on a practical basis a flexible general purpose digital computer. The subroutines given cover mainly a single level of organization of the machine (since few subroutines refer to other subroutines in the library) and include a wide range of operations that are basic to problems in analysis. It is to be regretted that they did not give more information on why they chose what they did and on the principles behind many of the subroutines so that the reader could more easily follow their path and adapt their methods to his own needs. *R. Hamming* (Murray Hill, N. J.).

Ansermet, A. Le calcul d'une paire d'ellipses d'erreur dont la forme est circulaire. Schweiz. Z. Vermessg. Kulturtech. 49, 200–207 (1951).

Bachmann, W. K. Sur la compensation des observations conditionnelles avec inconnues. Schweiz. Z. Vermessg. Kulturtech. 49, 190–200 (1951).

Kuntzmann, J. Notions de grille et de tube. Ann. Inst. Fourier Grenoble 2 (1950), 197–205 (1951).

A grill is a finite set of values of x, to each of which corresponds not only a value of $f(x)$ but also an interval of uncertainty. A tube is defined similarly except that the values and uncertainties are given over a continuous range of x. In most cases the magnitude of the uncertainty is assumed to be independent of x. If there exists a polynomial $P(x)$ of degree n whose values fall within the intervals of uncertainty of a given grill (or tube), then the latter is called a polynomial grill (or tube) of degree n, compatible with $P(x)$. A necessary and sufficient condition for a grill to be of degree n is that all the intervals of uncertainty of the nth divided differences have a common point. Practical methods of calculation, with examples, are given for determining whether or not a given grill or tube is of degree n, and for finding compatible polynomials.

The second part of the paper is concerned with enlarging a given grill of degree n by the insertion of a value and an interval of uncertainty at an intermediate point x (giving a new grill whose uncertainty intervals are no longer constant in magnitude) in such a way as to preserve compatibility with any polynomial of degree n with which the original grill is compatible. Such an insertion is called nonrestrictive because it imposes no additional restriction on a second insertion. Ordinary interpolation formulas may be used to calculate both the inserted value and the inserted interval. If a grill is only locally of degree n, an insertion at $x=a$ is called non-restrictive if the inserted interval contains all the intervals obtained by interpolating in the various subgrills of degree n which surround the point a.

P. W. Ketchum (Urbana, Ill.).

Curry, Haskell B. Note on iterations with convergence of higher degree. Quart. Appl. Math. 9, 204–205 (1951).

This note contains two remarks on a paper of E. Bodewig [same Quart. 7, 325–333 (1949); these Rev. 11, 136] concerning the convergence of a sequence in a degree at least m in the neighborhood of every root of a polynomial.

E. Frank (Chicago, Ill.).

Luke, Yudell L., and Ufford, Dolores. On the roots of algebraic equations. J. Math. Physics **30**, 94–101 (1951).

S. N. Lin [J. Math. Physics **20**, 231–242 (1941); **23**, 60–77 (1943); these Rev. **3**, 153; **5**, 49] gave a method for the determination of the zeros of a polynomial by a factorization of the given polynomial into two polynomials of lower degree. By a repetition of the process, the polynomial is finally factored into a set of polynomials whose zeros can be simply determined. In this paper, Lin's method is extended so that a given polynomial can be completely separated into any number of subpolynomials at the same time if the method converges. If the method is only partially convergent, methods are indicated for the determination of all the zeros. *E. Frank* (Chicago, Ill.).

Crandall, S. H. Iterative procedures related to relaxation methods for eigenvalue problems. Proc. Roy. Soc. London. Ser. A. **207**, 416–423 (1951).

The author discusses convergence properties of three iterative procedures for finding eigenvalues λ and eigenvectors X satisfying $AX=\lambda BX$, where A and B are $n\times n$ symmetric real matrices and B is positive definite. In all three methods, one starts with an initial vector V_0 and constructs a sequence of vectors V_k approximating to an eigenvector, using the recursion formula $(A-\Lambda_{k+1}B)V_{k+1}=R_{k+1}$. In the first method $R_{k+1}=R$ is held fixed and

$$\Lambda_{k+1}=(V_k'AV_k)/(V_k'BV_k)$$

is the Rayleigh quotient for V_k. In the second method $\Lambda_{k+1}=\Lambda$ is held fixed, while $R_{k+1}=BV_k$. In the third method both Λ_{k+1} and R_{k+1} are defined recursively in terms of V_k as above. *G. B. Thomas* (Cambridge, Mass.).

Maehly, H. J. Zur genäherten Berechnung der Eigenwerte einer Schrödinger-Gleichung. Nuovo Cimento (9) **8**, 466–468 (1951).

Recurrence formulas are quoted which enable one to calculate lower bounds for the eigenvalues of a time-independent Schrödinger equation. The lower bound for one eigenvalue must be known before lower bounds can be found for the next smaller eigenvalues. (Upper bounds can be found by the Ritz method.) *T. E. Hull.*

Lanczos, Cornelius. An iteration method for the solution of the eigenvalue problem of linear differential and integral operators. J. Research Nat. Bur. Standards **45**, 255–282 (1950).

Die Ausgangsfragestellung ist die Auffindung der Eigenwerte und Eigenvektoren bei einer n-reihigen Matrix. Der Kürze wegen sollen hier die Grundgedanken des Verfassers nur in dem einfachsten Fall besprochen werden, wenn die Matrix symmetrisch ist und keine mehrfachen Eigenwerte vorhanden sind. Es sei aber hervorgehoben, dass der Verfasser sehr ausführlich auch auf den allgemeinen Fall bei unsymmetrischer Matrix und mehrfachen Wurzeln eingeht und an illustrierenden Beispielen bespricht. Wendet man in dem hervorgehobenen Spezialfall auf einen willkürlich gewählten Vektor n-mal die der Matrix A entsprechende lineare Transformation an, so besteht zwischen den so erhaltenen $(n+1)$ Vektoren eine lineare homogene Relation, die in das charakteristische Polynom übergeht, wenn man den v-fach iterierten Vektor durch die $v$te Potenz einer Veränderlichen λ ersetzt (in unserem Fall identisch mit dem Hamilton-Cayley'schen Satz), und die Wurzeln dieser charakteristischen Gleichung sind dann die Eigenwerte. Für eine näherungsweise Bestimmung der Eigenwerte kann man sich mit einer geringeren Anzahl v $(v<n)$ von Iterationen begnügen und nach einer linearen Kombination fragen, bei der der Koeffizient des v-fach iterierten Vektors gleich 1 ist und bei der die übrigen Koeffizienten so bestimmt sind, dass das Quadrat des Absolutbetrages des durch die lineare Kombination dargestellten Vektors ein Minimum wird. Da die iterierten Vektoren aber bei grossem v annähernd parallel sind, würden bei einer direkten Ausführung einer derartigen Rechnung grosse Abrundungsfehler entstehen. Dies veranlasst den Verfasser, das Verfahren zu modifizieren. Er bezeichnet mit b_0' den durch Transformation mittels A aus b_0 entstehenden Vektor, und setzt an Stelle des dem Ausgangsvektor b_0 entsprechenden Vektor b_0' einen Vektor $b_1=b_0'-\alpha_0 b_0$, wobei α_0 durch die Forderung

$$(b_0'-\alpha_0 b_0)^2=\min.$$

bestimmt wird. In analoger Weise werden die mehrfach iterierten Vektoren durch geeignete lineare Kombination aus den bereits erhaltenen Vektoren ersetzt. Dabei erhält man für die b_v eine dreigliedrige Rekursionsformel. Der Verfasser nennt dieses Verfahren "minimalisierende Iteration". Als praktische Anwendung wird der einseitig eingeklemmte Stab behandelt, mit in beiden Hälften stückweise konstantem Trägheitsmoment. Durch Unterteilung des Intervalls in 12 Teile gelangt der Verfasser zu einem Eigenwertproblem bei einer zwölfreihigen Matrix und bei Verwendung von 6 Iterationen wird eine Genauigkeit bis auf 10 Stellen erreicht.

Die Resultate sind unabhängig von der Ordnung der Matrix und können daher auf Differential- und Integraloperatoren übertragen werden. Das Verfahren wird an numerischen Beispielen erläutert (Nullstellen der Besselfunktion nullter Ordnung, schwingende Saite und schwingender Stab). Am Schluss der Arbeit finden sich Erläuterungen zu einer ähnlichen Methode von Milne und ferner Andeutungen über die Erweiterung der Methode auf mehrdimensionale Probleme. *P. Funk* (Wien).

Arnoldi, W. E. The principle of minimized iteration in the solution of the matrix eigenvalue problem. Quart. Appl. Math. **9**, 17–29 (1951).

Der Verfasser knüpft seine Untersuchungen an folgende bekannte Methode zur Reduktion einer Matrix an: Seien $(\lambda I-u)k=0$, bzw. $\kappa(\lambda I-u)=0$, die, die Eigenvektoren k und κ definierenden Gleichungen. Ersetzt man nun k und κ durch $k=\sum_1^m c_i k_i$, $\kappa=\sum_1^m c_i\kappa_i$, so gelangt man durch Rechts- bzw. Linksmultiplikation von den rechteckigen Matrizen k_i und κ_i und nachträglicher Linksmultiplikation mit $[\kappa_j k_i]^{-1}$ zu einer Gleichung, die als die "reduzierte Gleichung" bezeichnet wird. Auf diese allgemeine Methode führt nun der Verf. sowohl die Methode von Lanczos, als auch die Methode von Galerkin zurück. Zur Methode von Lanczos [siehe das vorstehende Referat] gelangt man durch die Forderung $k_i\kappa_j=0$ $(i\neq j)$. Zur Methode von Galerkin [W. J. Duncan, Ministry of Aircraft Production, Aeronaut. Res. Committee, Rep. and Memoranda, no. **1798** (1937)] gelangt man, wenn man voraussetzt $\kappa_i=k_i$. Der Verfasser gibt dabei übersichtliche Vorschriften, wie die tatsächliche Rechnung durchzuführen ist und diskutiert den Rechenaufwand bei beiden Methoden. *P. Funk* (Wien).

van der Corput, J. G., and Franklin, Joel. Approximation of integrals by integration by parts. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 213–219 (1951).

Soit l'integrale: $I_0=\int_a^b g(x)e_0(f(x))dx$ où $g(x)$ est supposé complètement monotone (c'est à dire $(-1)^n g^{(n)}(x)\geqq 0$). Par

n integrations par parties, on obtient:

$$I_0 = u_0 - u_1 + \cdots + (-1)^{n-1}u_{n-1} + (-1)^n I_n$$

avec: $I_n = -\int_a^b g_n(x)f'(x)e_n(f(x))dx$ et

$$g_n(x) = g'_{n-1}(x)/f'(x), \quad de_{n+1}(t)/dt = e_n(t).$$

On obtient pour le dernier terme les majorations suivantes: si $e_n(t) \geqq 0$ $(n=0, 1, \cdots)$, $0 \leqq I_n \leqq u_n$; si $|e_n(f(x))| \leqq |e_n(f(a))|$ $(n=1, 2, \cdots)$, $|I_n| \leqq |u_{n-1}|$. Une application est donnée.
J. Kuntzmann (Grenoble).

van Heemert, A. On the numerical evaluation of certain types of integrals. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 55**, i+21 pp. (1949).

Discussion of various possible methods of evaluation for integrals of the form

$$\int_0^\pi r^{-1} \cos n\theta d\theta, \quad \int_0^\pi r^{-1}\theta \sin \theta d\theta, \quad \int_0^\pi r^{-3} \cos n\theta d\theta, \text{ etc.,}$$

where $r^2 = \cos^2\theta - 2\alpha \cos\theta + \beta$; $\alpha, \beta = \text{const}$. These integrals are of importance for the calculation of downwash fields of wings of various plan form. *E. Reissner.*

Herrick, Samuel. Step-by-step integration of $\ddot{x} = f(x, y, z, t)$ without a "corrector." Math. Tables and Other Aids to Computation **5**, 61–67 (1951).

L'auteur compare deux procédés d'intégration de

$$x'' = f(x, t).$$

Le premier utilise $\delta^2 x_n = k^2(x_n'' + \delta^2 x_n''/12 - \delta^4 x_n''/240)$ avec prédiction et correction (la prédiction est faite en supposant que $\delta^6 x'' = 0$). Le second utilise

$$x_n = k^2(\textstyle\sum_2 x_n'' + x_n''/12 - \delta^2 x_n''/240).$$

On peut utiliser des valeurs estimées beaucoup plus grossières de $\delta^2 x_n''$ et on peut se contenter de la prédiction (qui est faite en supposant $\delta^4 x'' = 0$). On peut donner des formules analogues à la seconde en utilisant des différences descendantes ou des valeurs antérieures de x''. *J. Kuntzmann.*

Nehari, Zeev. On the numerical computation of mapping functions by orthogonalization. Proc. Nat. Acad. Sci. U. S. A. **37**, 369–372 (1951).

Let D be a plane domain with an analytic boundary C, and let $K(z, \zeta) = \sum_{n=1}^\infty u_n(z)\overline{u_n(\zeta)}$ be the Szegő kernel function of D, of importance in the construction of conformal maps. Here the functions $\{u_n(z)\}$ are complete and orthonormal over C. The author gives an estimate of the truncation error in the computation of K. It is shown that

$$\left|K(z, \zeta) - \sum_{j=1}^n u_j(z)\overline{u_j(\zeta)}\right| \leqq s_n(z)s_n(\zeta)$$

where the functions s_n are as follows:

$$s_n(t) = (4\pi^2)^{-1}\int_C \frac{ds}{|z-t|^2} - \sum_{\nu=1}^n [|a_\nu(t)|^2 + |u_\nu(t)|^2],$$

$$a_\nu(t) = (2\pi)^{-1}\int_C \frac{\overline{u_\nu(z)}}{z-t} ds.$$

The functions $s_n(t) \to 0$ as $n \to \infty$, but since they involve the first n orthogonal functions, the truncation error may be estimated only after these latter functions have been determined. *P. Davis* (Cambridge, Mass.).

Timman, R. The direct and the inverse problem of aerofoil theory. A method to obtain numerical solutions. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 16**, 30 pp. (1 plate) (1951).

This report consists of a review of several previous reports in the same series [Greidanus and the author, **F. 10** (1947); the author, **F. 11**, **F. 12** (1947); **F. 13** (1948); J. Cohen, **F. 14** (1947); the author, **F. 32** (1948); these Rev. **11**, 404] together with some new work relating to the incompressible two-dimensional potential flow about simple airfoils. In addition, an elegant treatment is given of the role of complex function theory, conformal mapping and the theory of conjugate functions in the solution of problems of airfoil theory. The direct and inverse problems are discussed in detail and the results in the report are carefully compared with similar work done elsewhere. The direct and inverse problems are formulated as integral equations which are solved by iteration. The convergence of the iteration procedure for the inverse problem is established and it is suggested that the proof of the convergence of the iteration procedure for the direct problem can be carried out along the lines of Warschawski's proof of the convergence of the Theodorsen-Garrick method [Quart. Appl. Math. **3**, 12–28 (1945); these Rev. **6**, 207]. The second part of the report discusses in considerable detail the numerical evaluation of the Poisson integral and design procedures are given for certain types of airfoils. *C. Saltzer* (Cleveland, Ohio).

Wu, Chung-Hua, and Brown, Curtis A. Method of analysis for compressible flow past arbitrary turbomachine blades on general surface of revolution. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2407**, 42 pp. (1951).

The equation of the rotationally symmetric flow of an inviscid, compressible, isentropic fluid is formulated as a modified non-linear second order partial differential equation in two space variables for the stream function. The numerical solution of the associated finite difference equation is discussed by several methods including a matrix method, the relaxation method and the Liebman iteration procedure. A numerical example is given and the results are compared with experimental results. *C. Saltzer.*

Charney, J. G., Fjörtoft, R., and von Neumann, J. Numerical integration of the barotropic vorticity equation. Tellus **2**, 237–254 (1950).

As a first step in a program of weather prediction by the physical equations of meteorology, the barotropic vorticity equation was integrated numerically with the help of the ENIAC electronic computer at Aberdeen, Md. The vorticity equation becomes an elliptic partial differential equation, with the height change of an isobaric surface as dependent variable, when the wind is assumed to be geostrophic. The solution of the equation requires, aside from the knowledge of the initial contour field over the area considered, the height values around its boundary at all future times as well as the Laplacian of height along those portions of the boundary along which air enters the area. Since the boundary conditions affect only a relatively small part of the area during the forecast period, the height change as well as the change of the Laplacian with time were put equal to zero wherever their knowledge was required. Three hours were found to be a sufficiently short time interval for the successive numerical evaluation of the height change, with space grid points spaced 736 km apart. The resultant 24 hour forecasts agreed reasonably well with observations, and the

authors discuss the errors produced by the various assumptions. The 24 hour forecast took 24 hours to complete, a time which can be cut by a factor of two even with the ENIAC. With the almost completed Princeton computer the forecast time should become sufficiently short for the forecasts to develop very much faster than the weather itself. *H. Panofsky* (State College, Pa.).

Allen, D. N. de G., and Dennis, S. C. R. The application of relaxation methods to the solution of differential equations in three dimensions. I. Boundary value potential problems. Quart. J. Mech. Appl. Math. 4, 199–208 (1 plate) (1951).

Bei der direkten Berechnung der Lösung der Differenzengleichung werden die numerischen Werte in ein passend gezeichnetes kubisches Gitter eingetragen. Als Beispiele werden insbesondere behandelt: Würfel bei vorgegebener Oberflächentemperatur und das Quadrantenelektrometer. *P. Funk* (Wien).

***Salles, F., et Thorn, G. Méthode des différences finies appliquée aux problèmes bidimensionnels de calcul des contraintes d'une plaque.** Publ. Sci. Tech. Ministère de l'Air, Paris, Bull. Serv. Tech. no. 115, iv+85 pp. (1951).

An expository paper on applications of the method of finite differences to the determination of the Airy stress function ϕ satisfying the biharmonic equation $\nabla^4\phi=0$, subject to prescribed boundary values of ϕ, ϕ_x, and ϕ_y. Three numerical examples are treated in detail. The first is intended merely to illustrate the method; in the other two the results are compared with results obtained by other methods. The exposition is clear and is supplemented with numerous illustrations. There is a thorough discussion of how to proceed near a boundary or a corner of the region. *G. B. Thomas, Jr.* (Cambridge, Mass.).

Kuznecov, E. S., and Ovčinskiĭ, B. V. Results of numerical solution of the integral equation of the theory of the scattering of light in the atmosphere. Akad. Nauk SSSR. Trudy Geofiz. Inst., no. 4(131), 105 pp. (1949). (Russian)

This paper considers the problem of diffuse reflection and transmission by a plane-parallel atmosphere of optical thickness τ^* (>0) for the case of isotropic scattering with an albedo $0<1-q\leqq 1$ for single scattering. The underlying mathematical problem is that of solving the equation of transfer

$$(1)\quad \mu\frac{dI(\tau,\mu)}{d\tau}=I(\tau,\mu)-\tfrac{1}{2}(1-q)\int_{-1}^{+1}I(\tau,\mu')d\mu'-\tfrac{1}{4}(1-q)e^{-\tau/\mu_0},$$

together with the boundary conditions (2) $I(0,-\mu)=0$ and $I(\tau_1,+\mu)=0$ $(0<\mu\leqq 1)$. In (1) $0<\mu_0\leqq 1$ is the cosine of the angle of incidence of the incident parallel beam of light of unit flux. It is known that this problem is equivalent to that of solving the integral equation

$$(3)\quad \phi_q(\tau)=\tfrac{1}{4}e^{-(\tau^*-\tau)/\mu_0}+\tfrac{1}{2}q\mu_0e^{-\tau^*/\mu_0}E_2(\tau)+\int_0^{\tau^*}\phi_q(t)[\tfrac{1}{2}E_1(|t-\tau|)+qE_2(\tau)E_2(t)]dt,$$

where $E_n(x)$ denotes the exponential integral of order n.

As a preliminary to the solution of this problem the authors assemble in chapter I various known formulae involving the exponential integrals. In particular a list of the integrals

$$\int_0^x t^mE_n(t)dt,\quad \int_0^a t^mE_1(|x-t|)dt,\quad \int_0^x e^{kt}E_1(t)dt$$

is given; the last of these is a special case of the function denoted by $F_j(\tau,\mu)$ by H. van de Hulst, [Astrophys. J. 107, 220–246 (1948); these Rev. 10, 151, 855] and S. Chandrasekhar [ibid. 108, 92–111 (1948); these Rev. 10, 543]. A table of the first four exponential integrals for $x=0.01(0.01)0.6$ is also given.

In chapter II an iteration method of solving equation (3) is proposed. For this purpose, a quadrature formula for evaluating integrals of the form

$$(4)\quad I(\tau)=\int_0^{\tau^*}f(t)E_1(|t-\tau|)dt,$$

is constructed. Thus writing

$$\tau=kh\quad\text{and}\quad t=mh\quad(k,m=0,\cdots,n)$$

where h is a suitably chosen small fraction, we can express the integral (4) in the form

$$I(kh)=\sum_{m=0}^{n-1}[P_{km}f(mh)+Q_{km}h^{-1}\Delta f(mh)],$$

where

$$P_{km}=E_2[(m-k)h]-E_2[(m-k+1)h],$$
$$Q_{km}=E_3[(m-k)h]-E_3[(m-k+1)h]-hE_2[(m-k+1)h],$$

and $\Delta f(mh)=f[(m+1)h]-f(mh)$. The quantities P_{km} and Q_{km} satisfy a number of recurrence relations. For convenience of calculation for $\tau^*\leqq 0.6$, the quantities $P_{0,m}$, $P_{60,m}$, $Q_{0,m}$ and $Q_{60,m}$ $(m=0,1,\cdots,60)$ and $P_{0,m}$, $P_{30,m}$, $Q_{0,m}$ and $Q_{30,m}$ $(m=0,1,\cdots,30)$ appropriate for $h=0.01$ and 0.02, respectively, are tabulated.

For the case $q=0$ the iteration itself is performed by correcting an initial approximate solution $\bar{\phi}(\tau)$ in the manner $\phi_0(\tau)=\bar{\phi}(\tau)+A+B\tau+C\tau^2$ and determining the constants A, B and C by satisfying the integral equation at $\tau=0$, $\tau=\tau^*/2$ and $\tau=\tau^*$. Solutions obtained in this manner are tabulated for $\tau^*=0.2$, 0.3, 0.4, 0.5 and 0.6 for $\cos^{-1}\mu_0=30°$, 45°, 60° and 75°. For the case $q>0$, the solution is expressed in the form $\phi_q(\tau)=\phi_0(\tau)+H\omega(\tau)$, where

$$H=q\frac{\mu_0e^{-\tau^*/\mu_0}+2\int_0^{\tau^*}\phi_0(t)E_2(t)dt}{1-2q\int_0^{\tau^*}\omega(t)E_2(t)dt},$$

and $\omega(\tau)$ is a solution of the equation

$$\omega(\tau)=\tfrac{1}{2}E_2(\tau)+\frac{1}{2}\int_0^{\tau^*}\omega(t)E_1(|t-\tau|)dt.$$

An approximate solution of this last equation (as indicated by a direct solution of the equation of transfer) is $\omega(\tau)=(\tau^*-\tau+\frac{1}{2})/(\tau^*+1)$; a higher approximation is obtained by the authors by iterating this solution. The coefficient H is tabulated for $\tau^*=0.2$, 0.3, 0.4, 0.5 and 0.6 for $\cos^{-1}\mu_0=30°$, 45°, 60° and 75°. For these same arguments the functions $\phi_q(\tau)$ are tabulated for $q=0.1$, 0.2, 0.3 and 0.8.

Solutions obtained in the manner described above are compared with the solutions obtained by the standard methods of solving the equation of transfer itself. In particular, a detailed comparison is made with the solutions obtained in the second approximation according to the method

described by Chandrasekhar [Astrophys. J. **101**, 348–355 (1945); **103**, 165–192 (1946); these Rev. **6**, 244; **7**, 489]. [It should be stated that exact solution for the problem considered by these authors is known and that results of greater accuracy are obtained by the method described in S. Chandrasekhar, Radiative Transfer, Oxford, 1950, see particularly §§62 and 63; these Rev. **13**, 136.]

S. Chandrasekhar (Williams Bay, Wis.).

Polisar, G. L. Electrical method for the solution of a certain problem in dynamics of the aeroplane. Tehnika Vozdušnogo Flota **1947**, no. 7, 15–18 (1947). (Russian)

The author briefly describes a differential analyzer for solution of up to 6 linear first order differential equations with constant coefficients. Constructed at the Power Institute of the Academy of Sciences, its basic design, which uses conventional capacity and amplifier integrating circuits, is attributed to L. I. Gutenmaher [C. R. (Doklady) Acad. Sci. URSS **47**, 259–262 (1945); these Rev. **7**, 221]. Errors not greater than the width of the output as shown on an oscillograph are claimed for its solution of a test problem. The behavior of an actual autopilot is investigated by connecting it to the differential analyzer as a model of the plane to be controlled by it. *R. Church* (Monterey, Calif.).

Polisar, G. L. Investigation of non-linear systems of automatic regulation by the method of combination of objects to be tested with an electro-integrator. Izvestya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 384–395 (1949). (Russian)

When the work here reported was undertaken the author had at his disposal an electronic differential analyzer, apparently similar to that described in the preceding review, able to handle linear differential equations of eighth order and capable of introducing simple non-linearities arising in problems of regulation. No description of this equipment is given but the oscillogram results of some test problems indicate errors not exceeding 1.5%. In this paper the principal attention is given to the now much used method of partial simulation suggested in the paper reviewed above according to which a part of a system to be studied is replaced by an electronic differential analyzer (or appropriate equivalent) which is connected through suitable transducers to the rest (e.g., controller) of the system or a small-scale model of it. Investigation of a system for automatic temperature regulation of an airplane engine by this method gave results in close agreement with those furnished by full-scale experimentation that were available. The numerical solution of a set of differential equations for this entire regulation system had given less satisfactory agreement.

R. Church (Monterey, Calif).

Polisar, G. L. Investigation of the dynamical behavior of complicated systems by combining calculating apparatus with objects not thus replaced by models. Doklady Akad. Nauk SSSR (N.S.) **61**, 641–644 (1948). (Russian)

Brief report on work described in the two preceding reviews. *R. Church* (Annapolis, Md.).

Polisar, G. L. The synthesis of mathematical machines and real objects. Doklady Akad. Nauk SSSR (N.S.) **74**, 711–714 (1950). (Russian)

The method of partial simulation is here considered when the simulator is a high speed digital computer. Attention is drawn to the errors arising from the circumstance that the computer and unreplaced equipment are interconnected only at a succession of discrete moments between which, for the situation studied here, each proceeds independently from initial conditions furnished by the other. As an example these errors are estimated and their compensation considered for a feed-back system governed by the equations $du_1/dt=u_2-u_1$ and $du_2/dt=u_1-u_2$. *R. Church.*

Kerner, Edward H. The solution of the Schrödinger equation for an approximate atomic field. Physical Rev. (2) **83**, 71–75 (1951).

The atomic potential with $Z/(1+Ar)$, where A is a parameter, as the effective nuclear charge is discussed and is shown to approximate reasonably well to Hartree or Fermi-Thomas potentials. The corresponding Schrödinger equation leads to a confluent form of the Huen equation and the solution is found approximately by expanding in hydrogenic functions; the eigenvalues are the roots of an infinite determinant and can be calculated with arbitrary accuracy. Numerical results are given for the lower states of mercury and for the scattering amplitude of a fast electron.

T. E. Hull (Vancouver, B. C.).

Gossard, Myron L. An iterative transformation procedure for numerical solution of flutter and similar characteristic-value problems. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2346**, 75 pp. (1951).

The bending vibrations of a cantilever beam satisfy the equation (1) $y=\omega^2\int_0^x dx\int_0^x (EI)^{-1}dx\int_x^L dx\int_x^L \gamma y dx$, ω frequency, γ mass per unit length, EI stiffness. The first mode y_1 and frequency ω_1 are easily found by iteration. To find the second mode, choose a mode $y_{a2}^{(1)}$ with zero ordinate at some station A. Substitution of $y_{a2}^{(1)}$ in the right hand side of (1) and integration gives a transformed mode $y_b^{(1)}$. A multiple of the first mode y_1 is subtracted from $y_b^{(1)}$ making the resulting displacement zero at station A, i.e., $y_{a2}^{(2)}=y_b-(y_b/y_1)_A y_1$. Then the process is repeated with $y_{a2}^{(2)}$ instead of $y_{a2}^{(1)}$ etc. until the ratio $y_{a2}^{(n+1)}/y_{a2}^{(n)}$ becomes a constant along the span. This constant is the square of the second frequency ω_2^2 and the second mode is found from $y_{a2}^{(n+1)}$. The higher modes and frequency can be calculated in a similar manner. The method is also applied to coupled bending-torsion vibrations and to flutter. *W. H. Muller* (Amsterdam).

Shinbrot, Marvin. A least squares curve fitting method with applications to the calculation of stability coefficients from transient-response data. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2341**, 52 pp. (1951).

Assuming for the response of an airplane to a disturbance $F(t)$ the solution of a differential equation

$$(1)\qquad P_0(D)q(t)=P_1(D)F(t),$$

where $D=d/dt$, $P_0=\sum_{i=0}^n a_i D^{n-i}$, $P_1=\sum_{k=0}^m C_k D^{m-k}$, it is required to find the coefficients a_i and C_k from the measured response to a known transient disturbance, i.e. from a set of values $q_m(t_\nu)$, $\nu=1, 2, \cdots, s$. Consider the set of solutions of (1) obtained by varying the constants a_i and C_k and let $q_e(t)$ be the solution for which $M=\sum_{\nu=1}^s[q(t_\nu)-q_m(t_\nu)]^2$ is a minimum. Then $q_e(t)$ corresponds to the desired values of a_i and C_k. By expanding the general solution of (1), $q(t)=\sum_{j=1}^n \varphi_j(B_r, \lambda_k, t)e^{\lambda_j t}$ (λ_j roots of $P_0(\lambda)=0$, B_r functions of C_k), in a Taylor's series in the neighbourhood of an approximation $B_r^{(0)}$, $\lambda_k^{(0)}$ and retaining only the first order terms, the problem is linearized. Substituting in M and minimizing gives linear equations for the increments of B_r and λ_k. Repeating this procedure, B_r and λ_k are found and from these the constants a_i and C_k. Examples are given with

$n=2$ and $m=1$ and for $F(t)$ a pulse, a step-function and a damped oscillation. *W. H. Muller* (Amsterdam).

Woodward, P. M. Time and frequency uncertainty in waveform analysis. Philos. Mag. (7) 42, 883–891 (1951).

Kolscher, Max. Harmonische Analyse einer gestörten Schwingung nach Amplitudenbegrenzung. Arch. Elektr. Übertragung 5, 293–299 (1951).

Given a periodic function u of the time t. Let a filter operate to replace each ordinate by a function thereof, viz., $i=f(u)$, not involving the time explicitly, and independent of preceding or succeeding values of u. The representation of i obtainable by expressing f as a power series is often found to fall short of meeting practical needs on account of poor convergence of the series for many of the more useful filters. The author proposes another procedure, involving a function of two variables derived from u. Let U be a new variable on the same range as u. Define $v(U,t)=1$ $(U\leqq u(t))$; $=0$ $(U>u(t))$. The harmonic coefficients of i are found to be inner products of the harmonic coefficients of v by the derivative of f. This representation makes it easy to see what the effect of a proposed filter f will be in the case of functions v for which the harmonic analysis can be carried through without too much labor. The author works out various numerical results for functions u expressible as the sum of two sinusoids. *A. Blake* (Buffalo, N. Y.).

Artobolevskiĭ, I. I. A mechanism for the solution of quadratic equations of the form $x^2-px+q=0$. Doklady Akad. Nauk SSSR (N.S.) 79, 401–403 (1951). (Russian)

Let $KCDN$ be a quadrangle with right angles at K and N, $KC=1/p$, $DN=q/p$. Choose M on CD so that $\angle KMN=\frac{1}{2}\pi$. Then KM intersects ND at A so that $NA=x$ is a root of $x^2-px+q=0$. The paper gives two routine mechanizations of this construction by means of rigid bars and crossheads. *A. W. Wundheiler* (Chicago, Ill.).

Barnes, R. C. M., Cooke-Yarborough, E. H., and Thomas, D. G. A. An electronic digital computor using cold cathode counting tubes for storage. I. Electronic Engrg. 23, 286–291 (1951).

ASTRONOMY

Ghosh, N. L. Equilibrium of rotating fluid-bodies in confocal stratifications. I, II. Bull. Calcutta Math. Soc. 42, 227–241, 242–248 (1950).

On making use of the potential of a heterogeneous spheroid with stratifications confocal to the boundary in terms of elliptic coordinates the author regains the results of P. Dive [Rotations internes des astres fluides, Blanchard, Paris, 1930]. Other investigations on the subject of the controversy between the results of P. Dive and A. Véronnet [see, for example, P. Appell, Traité de mécanique rationelle, tome 4, fasc. 2, 2d ed., Gauthier-Villars, Paris, 1937] are not mentioned. The reviewer was not able to verify certain points of the discussion. A special case of the law of density is worked out fully. *W. Jardetsky* (New York, N. Y.).

Bragard, L. Sur la relation fondamentale de la géodésie dynamique. Bull. Soc. Roy. Sci. Liège 19, 475–478 (1950).

Bragard, L. Sur la détermination de la figure d'équilibre d'une masse fluide en rotation uniforme. Bull. Soc. Roy. Sci. Liège 19, 479–484 (1950).

On assuming that the configuration of a fluid mass rotating with constant angular velocity is given in the form $s=a_m[1+\sum_{i=1}^{l}A_iX_{2i}+\sum_{n=0}^{\infty}u_n]$ and the gravity at this surface is $g=G_m[1+\sum_{i=1}^{l}B_iX_{2i}+\sum_{n=0}^{\infty}v_n]$ it is proved that $v_1=0$, $v_n=(n-1)u_n$, $n=2,3,\cdots$. This equation is called the basic relationship of dynamical geodesy. [See also the author, Institut Royal Colonial Belge. Sect. Sci. Tech. Mém. Coll. in 4°. 5, no. 1 (1949).] *W. S. Jardetzky.*

Jeffreys, Harold. Dynamic effects of a liquid core. II. Monthly Not. Roy. Astr. Soc. 110, 460–466 (1950).

L'auteur a étudié antérieurement [I: même journal 108, 206–209 (1948); II: ibid. 109, 670–687 (1949); ces Rev. 11, 745] la nutation eulérienne et la nutation astronomique d'une Terre composée d'un noyau liquide homogène enfermé dans une coque (I) indéformable et (II) élastique. Si la densité du noyau est supposée variable, la gravité doit diminuer ses mouvements. L'auteur étudie le cas d'un noyau incompressible dans une coque indéformable. Sa solution, analogue à celle de II, ne satisfait plus aux équations du mouvement, et permet seulement une estimation des périodes. En utilisant la loi de densité de Bullen, corrigée d'une erreur numérique, les modifications par rapport à I apparaissent négligeables, grâce au fait qu'on avait utilisé un noyau homogène ayant les moments d'inertie (et non la masse) du noyau réel. Si on tient compte enfin de l'élasticité de la coque par comparaison entre I et II, l'accord avec les observations récentes est satisfaisant. *J. Coulomb.*

Jeffreys, Harold, and Bland, Merriell E. M. The instability of a fluid sphere heated within. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. 6, 148–158 (1951).

G. F. S. Hills, et aussi Vening Meinesz (non cité par les auteurs) ont attribué la répartition des grands continents à des courants de convection cellulaire au cours du refroidissement de la Terre. Pour en faire la théorie, les auteurs supposent que la Terre est un fluide incompressible, que les différences de densité y sont faibles, que le flux de chaleur et la température superficielle sont constants. L'écart V de la température par rapport à l'état stationnaire est alors solution d'une équation du sixième ordre analogue à celle du problème plan classique. La discussion dépend du nombre $\lambda=\alpha Ph^5g/3\kappa^2\nu$, où α est le coefficient de dilatation cubique, h le rayon de la sphère, g l'accélération de la pesanteur, κ la conductivité thermique, ν la viscosité cinématique, et enfin P est la vitesse avec laquelle croîtrait la température en l'absence de convection, en sorte que la convection est plus facile si λ est plus faible. λ_n correspondant au cas où V contient un harmonique sphérique d'ordre n, on trouve approximativement $\lambda_1=3200$, $\lambda_2=5640$, $\lambda_3=9780$, et rigoureusement $\lambda_1=3090$. Ceci serait en faveur d'un courant diamétral et pourrait expliquer la concentration des continents sur un hémisphère. Mais de nombreuses difficultés (avant tout la présence du noyau terrestre) diminuent l'importance géophysique de ce résultat. *J. Coulomb.*

Agostinelli, Cataldo. Effetti sulla rotazione della Terra di una legge di attrazione analoga a quella di Weber. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 84, 48–62 (1950).

Following up his previous work [Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 167–189 (1949); these Rev. 11,

407] on the results of assuming a potential of the form $(1+\dot{r}^2/c^2)kmm'/r$ where c is the velocity of light, the author finds that there is no secular term in the angle of nutation and the angular velocity but that for the Earth this potential gives rise to an annual increment in the angle of precession of approximately 2.16″. *R. G. Langebartel.*

Carrus, Pierre A., Fox, Phyllis A., Haas, Felix, and Kopal, Zdeněk. The propagation of shock waves in a stellar model with continuous density distribution. Astrophys. J. 113, 496–518 (1951).

It is shown that in a perfect gas with constant specific heats the attenuation of a spherical shock with distance r from the centre is completely halted (i.e. its Mach number remains constant) if it is moving into a medium whose density diminishes as $r^{-4/3}$ (so that, in order that it be in equilibrium under its own gravitational field, the pressure diminishes as r^{-6}). (Compare surface waves moving into shallow water.) In fact the authors find a solution of the equations of motion with spherical symmetry in which the pressure, density and velocity behind the shock depend only on $\xi=r^{-5}t^4$. The shock radius therefore increases as $t^{4/5}$. An inner core becomes evacuated and its radius also increases as $t^{4/5}$. It is shown that the work described constitutes the general case in which the equations of motion reduce to ordinary differential equations in a single variable ξ. The equations are solved numerically for adiabatic index $\gamma=5/3$, and for the values $\sqrt{3}$, $\sqrt{10}$, $\sqrt{20}$, $\sqrt{50}$, $\sqrt{100}$, $\sqrt{200}$, $\sqrt{500}$ and ∞ for the shock Mach number, and the results expressed in the form of convenient tables and graphs. It is suggested that the present model may bear some relation to the earlier stages of a nova explosion, and the authors' previous work, on the generalized Roche model [same J. 113, 193–209 (1951); these Rev. 12, 643], to the later stages.

M. J. Lighthill (Manchester).

Unsöld, A. Cosmic radiation and cosmic magnetic fields. I. Origin and propagation of cosmic rays in our galaxy. Physical Rev. (2) 82, 857–863 (1951).

The author suggests that both cosmic rays and the radio emission from the galaxy originate in a special class of stars which are permanently "disturbed". *S. Chandrasekhar.*

Biermann, Ludwig, and Schlüter, Arnulf. Cosmic radiation and cosmic magnetic fields. II. Origin of cosmic magnetic fields. Physical Rev. (2) 82, 863–868 (1951).

This paper discusses the problem of cosmic magnetic fields in essentially the same terms as in an earlier paper of the authors [Z. Naturforschung 5a, 237–251 (1950); these Rev. 12, 291]. *S. Chandrasekhar* (Williams Bay, Wis.).

Menzel, Donald H., and Sen, Hari K. Transfer of radiation. II. Radiative transfer in absorption lines. Astrophys. J. 113, 482–489 (1951).

The method described by the authors in an earlier paper [same J. 110, 1–11 (1949); these Rev. 11, 185] is applied to the Milne-Eddington problem in the theory of the formation of absorption lines in stellar atmospheres. The expressions for the emergent intensities are numerically evaluated and compared with the values given by the exact solution for the problem obtained by Chandrasekhar [same J. 106, 145–151 (1947); these Rev. 9, 444].

S. Chandrasekhar (Williams Bay, Wis.).

Menzel, Donald H., and Sen, Hari K. Transfer of radiation. III. Reflection effect in eclipsing binaries. Astrophys. J. 113, 490–495 (1951).

In this paper the authors apply their method [see preceding review] to the problem of diffuse reflection by a semi-infinite atmosphere under conditions of isotropic scattering. Comparison with the known exact solution [S. Chandrasekhar, Radiative Transfer, Oxford, 1950, section 42, p. 134; these Rev. 13, 136] shows that the expansion of the source function in terms of exponential integrals is not as satisfactory in this case as in the other problems examined.

S. Chandrasekhar (Williams Bay, Wis.).

Hitotuyanagi, Zyuiti. Zur Theorie der Fraunhoferschen Linien in der Sonnenatmosphäre. Jap. J. Astr. Geophysics 21, 87–123 (1945).

In the equation of transfer appropriate for line formation in stellar atmosphere the ratio η of the line (l) to the continuous (k) absorption coefficients occurs. In the solution of the transfer problem some assumption concerning the variation of η with depth is generally made. For the case $\eta=$constant the exact solution for the emergent radiation is known [cf. S. Chandrasekhar, Radiative Transfer, Oxford, 1950, p. 321; these Rev. 13, 136]. And for the case when the departures from constancy of η can be considered small, Strömgren [Astrophys. J. 86, 1–27 (1937); see also M. Tuberg, ibid. 103, 145–164 (1946)] developed a perturbation theory and showed how in the so-called Milne-Eddington approximation the formula for the emergent flux of the radiation can be expressed in the same form as when η is considered constant provided $\lambda=1/(1+\eta)$ and $\lambda^{\frac{1}{2}}$ which occur in that formula are interpreted as mean values suitably defined. In this paper the author extends the perturbation theory to obtain the complete angular distribution of the emergent radiation. The expression for the emergent radiation, the author obtains, involves in addition to the two kinds of averages which Strömgren introduced, a third kind of average depending on the angle of emergence considered. The author compares his expression with the explicit solutions (in the Milne-Eddington approximation) known for certain analytic variations of η and shows that the perturbation theory gives an adequate approximation. The theory is used to analyze some observational data on the variation of line contours over the solar disc.

S. Chandrasekhar (Williams Bay, Wis.).

RELATIVITY

Becquerel, Jean. Remarques sur le ralentissement du cours du temps par l'effet d'un champ de gravitation. C. R. Acad. Sci. Paris 232, 1617–1619 (1951).

Applying Newtonian gravitation to the mass of a photon, one obtains a gravitational red shift formula which is in good agreement with the relativistic formula. The author points out that one cannot use this result as an argument against general relativity. The same red shift effect, when interpreted on the basis of the wave aspect of light, shows that space-time cannot be Euclidean. This agrees, at least qualitatively, with the point of view of the general theory of relativity. *A. Schild* (Pittsburgh, Pa.).

Vaidya, P. C. Nonstatic solutions of Einstein's field equations for spheres of fluids radiating energy. Physical Rev. (2) **83**, 10–17 (1951).

The author's summary is as follows: The energy tensor for a mixture of matter and outflowing radiation is derived, and a set of equations following from Einstein's field equations are written down whose solutions would represent nonstatic radiating spherical distributions. A few explicit analytical solutions are obtained, which describe a distribution of matter and outflowing radiation for $r \leqq a(t)$, an ever-expanding zone of pure radiation for $a(t) \leqq r \leqq b(t)$ and empty space beyond $r = b(t)$. Since $db(t)/dt$ is almost equal to 1 and $da(t)/dt$ is negative, the solutions obtained represent contracting distributions, but the contraction is not gravitational because m/r is a constant on the boundary $r = a(t)$, m being the mass. The contraction is a purely relativistic effect, the corresponding newtonian distributions being equilibrium distributions. It is hoped that the scheme developed here will be useful in working out solutions which would help in a clear understanding of the initial or the final stages of stellar evolution. *A. E. Schild* (Pittsburgh, Pa.).

Udeschini, Paolo. Le equazioni di seconda approssimazione nella nuova teoria relativistica unitaria di Einstein. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 21–24 (1951).

Udeschini, Paolo. Le equazioni di seconda approssimazione nella nuova teoria relativistica unitaria di Einstein. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 121–123 (1951).

In an earlier paper [same journal (8) **9**, 256–261 (1950); these Rev. **12**, 864] the author considered the field-equations of the Einstein unified theory to a first approximation. He now proceeds to a second approximation and shows that the gravitational and electromagnetic fields are no longer disjoint, but are inextricably fused together in a single field. In particular, the velocity of light is affected by the electromagnetic as well as by the gravitational field. The case of a static field is worked out in some detail, and it is found that, whereas in general relativity the velocity of light V satisfies (in empty space) the 3-dimensional Laplace equation, in the new theory it satisfies (in the second approximation) a Poisson equation of which the right-hand side is an invariant depending upon the magnetic field. *H. S. Ruse* (Leeds).

Udeschini, Paolo. Sulle mutue azioni fra campo gravitazionale e campo elettromagnetico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 390–394 (1951).

The author takes the asymmetric fundamental tensor of the Einstein unified field-theory to be $g_{\iota\kappa} = a_{\iota\kappa} + b_{\iota\kappa} + c_{\iota\kappa}$, where the $a_{\iota\kappa}$ (symmetric) have galilean values, the $b_{\iota\kappa}$ are small of the first order and the $c_{\iota\kappa}$ are small of the second order. The gravitational and electromagnetic fields are respectively specified by the symmetric and skew-symmetric parts of $g_{\iota\kappa}$, and therefore, to a first and second approximation, respectively, by the symmetric and skew-symmetric parts of $b_{\iota\kappa}$ and $c_{\iota\kappa}$. The present paper follows others by the author. In the paper cited in the preceding review he showed that, if the $c_{\iota\kappa}$ are neglected, the gravitational and electromagnetic fields separate out, and in the papers reviewed above that, if the $c_{\iota\kappa}$ are retained, the fields are in general indissolubly linked through the field-equations. Schrödinger [Communications Dublin Inst. Advanced Studies. Ser. A. no. 6 (1951); these Rev. **12**, 757] has examined the case when $b_{\iota\kappa}$ is skew and $c_{\iota\kappa}$ symmetric, the electromagnetic and gravitational fields thereby being assumed respectively of the first and second order. The author obtains Schrödinger's equations from his own more general ones, and shows that the first-order electromagnetic field can generate a second-order gravitational field. He then considers the dual case when the gravitational field is of the first order ($b_{\iota\kappa}$ symmetric) and the electromagnetic field is of the second order ($c_{\iota\kappa}$ skew), and deduces that, in contradistinction to the Schrödinger case, the first-order gravitational field has no influence on the second-order electromagnetic field, the equations of the former being separate from those of the latter. *H. S. Ruse* (Leeds).

Scheidegger, Adrian E. Zum Zusammenhang zwischen Feld- und Bewegungsgleichungen. Helvetica Phys. Acta **23**, 740–744 (1950).

In a paper by Einstein, Infeld and Hoffman [Ann. of Math. (2) **39**, 65–100 (1938)], the equations of motion in relativity theory were deduced from the field equations. The procedure described there was generalized by Bergmann [Physical Rev. (2) **75**, 680–685 (1949); these Rev. **10**, 408]. The author of the article under review asks whether the equations of motion can be deduced from a non-linear electrodynamics like that of Born and Infeld. The answer is that this is not possible and that a field theory from which the equations of motion can be deduced must be at least a tensor theory. *L. Infeld* (Warsaw).

Infeld, L., and Scheidegger, A. E. Radiation and gravitational equations of motion. Canadian J. Math. **3**, 195–207 (1951).

Unlike those of most other field theories, the field equations in relativity theory are non-linear. This implies many difficulties in defining the notion of wave and so on. In other respects, however, close analogies between relativity theory and other classical field theories exist. It is the object of this paper to investigate such analogies. Some time ago Bergmann and Brunings showed [Rev. Modern Physics **21**, 480–487 (1949); these Rev. **11**, 299] that the coordinate system can be chosen so that the equations of motion have any form we wish. Especially, a coordinate system can be found in which the motion is simply Newtonian. In such a system, however, the metric is very complicated. The authors are led to a new version of the usual approximation method: they can enforce integrability by the simple procedure of changing the coordinate system. The authors investigate whether the general solution γ^* of the field equations calculated by Einstein and Infeld [Canadian J. Math. **1**, 209–241 (1949); these Rev. **11**, 59] can be obtained from the particular γ's by a coordinate transformation. This is true. Some of the terms in the power series for $\gamma_{\alpha\beta}$ are analogous to the ones representing radiation in electromagnetic theory. The authors adopt, therefore, the name "radiation terms" for those terms. It can be proved that the term starting the radiation expansion is of just such form that it can be created by a coordinate transformation. The result is that if "radiation terms" are added at a certain stage of the approximation they are either meaningless or make the approximation procedure inconsistent, conclusions which can be drawn from the statement that it is always possible to set up such a coordinate system that the relativistic equations of motion of any order have Newtonian form. The metric in this case is by no means of Newtonian character near the singularities. It is only a matter of representation whether the rela-

tivistic effects are explicitly contained in the equations of motion or in the metric field. The authors illustrate this by a specific example discussing some results of H. P. Robertson [Ann. of Math. (2) **39**, 101–104 (1938)] and A. E. Scheidegger [Thesis, University of Toronto, 1950]. They obtain either simple (Newtonian) equations of motion and a complicated metric field, or a simple field (of Newtonian character near the singularities) but non-Newtonian equations of motion. *M. Pinl* (Dacca).

√***Scheidegger, Adrian E. On gravitational radiation.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 218–224. University of Toronto Press, Toronto, 1951. $6.00.

Discussing the results of some papers of A. Einstein, L. Infeld, B. Hoffmann and P. R. Wallace [cf. A. Einstein and L. Infeld, Canadian J. Math. **1**, 209–241 (1949); N. Hu, Proc. Roy. Irish Acad. Sect. A. **51**, 87–111 (1947); L. Infeld and P. R. Wallace, Physical Rev. (2) **57**, 797–806 (1940); these Rev. **11**, 59; **8**, 496; **1**, 274; A. Einstein, L. Infeld and B. Hoffmann, Ann. of Math. (2) **39**, 65–100 (1938); L. Infeld, Physical Rev. (2) **53**, 836–841 (1938)] in a detailed manner, the author reaches the following conclusions: Infeld made no contribution to the equations of motion in the gravitational field because he did not carry his calculations far enough, and Hu obtained a strange result because he used an odd coordinate system. A suitable coordinate transformation would transform Hu's metric into the usual one and annihilate his result. Moving masses have no radiation damping connected with them as have moving charged particles. Two moving bodies, such as double stars, are in a secularly stable movement, they do not radiate gravitational waves and there is no slowing down of the motion by a radiation damping force. The non-linear theory of relativity leads to entirely different effects than one would like to expect by analogy with the linear theory of electrodynamics. But if it is possible to solve the gravitational field equations by a method entirely different from the quasi-stationary approach of Einstein, Infeld and Hoffmann, then there may still exist a non-linear phenomenon which one could call a gravitational wave. *M. Pinl* (Dacca).

Scheidegger, Adrian E. Gravitational transverse-transverse waves. Physical Rev. (2) **82**, 883–885 (1951).

Infeld and the present author have stated in several earlier papers [see the two preceding reviews] that no systems of masses subject to their gravitational field alone can radiate gravitational waves if the approximation procedure of these authors is adopted. This result was proved explicitly only for radiation of purely longitudinal and transverse-longitudinal waves. The author gives first of all a new deduction of that fact for the types of waves just mentioned. Furthermore the case of transverse-transverse waves is investigated and it is shown explicitly that there cannot be any radiation of such waves either. It may well be that solutions of Einstein's field equations exist, which describe a wave, but those solutions cannot be obtained by the approximation procedure which is under consideration here. The author compares these facts with the similar situation in electromagnetic theory regarding the so-called retarded Lienard-Wiechert potentials. *M. Pinl* (Dacca).

Ivanenko, D. D., and Brodskiĭ, A. M. Gravitational radiation damping. Doklady Akad. Nauk SSSR (N.S.) **75**, 519–522 (1950). (Russian)

Using the linear-approximation gravitational field equations, the authors obtain the gravitational potentials (or deviations of the metric tensor components from their Galilean values) of a moving particle, in the form of one half the difference between the advanced and retarded potentials, expanded in powers of $1/c$. On substituting these quantities into the equations of motion of the particle (the equations of a geodesic), they find terms in the equations that can be interpreted as describing gravitational radiation damping.

However, it should be pointed out that such a procedure, based on the linear approximation of the gravitational equations, is doubtful. This is due, among other things, to the fact that, in the exact equations, the integrability conditions for the higher order terms impose restrictions on the terms of lower order. In fact, it appears that a system of masses subject only to their gravitational field cannot radiate gravitational waves. See the paper by A. E. Scheidegger reviewed above; references to other papers on this question by Infeld and Scheidegger will be found there. *N. Rosen* (Chapel Hill, N. C.).

Brdička, M. On gravitational waves. Proc. Roy. Irish Acad. Sect. A. **54**, 137–142 (1951).

The author obtains a solution of the gravitational (general relativity) and electromagnetic field equations corresponding to a plane wave, the electromagnetic part being plane polarized and the gravitational part being of the transverse-transverse type. However, to get such a solution without singularities, the energy-momentum density tensor of the electromagnetic field was taken to be the negative of the Maxwell tensor. Hence the solution, which the author points out had been previously encountered by A. Papapetrou [same Proc. Sect. A. **51**, 191–204 (1947); these Rev. **10**, 157], has no physical significance, but is of some interest in indicating the role played by the gravitational energy-momentum pseudo-tensor in the determination of the metric. *N. Rosen* (Chapel Hill, N. C.).

MECHANICS

Geronimus, Ya. L. An application of the method of best approximation to balancing. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **4**, no. 14, 64–67 (1948). (Russian)

For a rotating crankshaft (α is crank angle), the author introduces complex numbers in the plane perpendicular to its axis. If p and m_{xy} are then the resultant vector and moment after application of counterbalances, the best mean balancing occurs when $\int|p|^2d\alpha$ and $\int|m_{xy}|d\alpha$ are simultaneously minima. The best uniform balancing occurs when $\max|p|$ and $\max|m_{xy}|$ are minima. For two counterbalances the resulting equations can be solved with ease in terms of centers and centers of gravity of certain representative points of the complex plane. *A. W. Wundheiler.*

Geronimus, Ya. L. On the calculation of the counterweights of a crankshaft to reduce the bearing loads. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **2**, 164–174 (1947). (Russian)

The method of the preceding paper is applied to the problem of bearing loads. The mean-square reactions ($\int_0^{4\pi}K^2d\alpha/4\pi$) are minimized. It is found that the counterweights should

balance the load component that rotates with the crankshaft. The problem is reduced to a set of linear complex equations. For a six-throw crankshaft, the equations are solved for four and six counterweights. For a four-throw crankshaft a set of three weights is computed. A numerical illustration is given for the Curtiss Conqueror engine.
A. W. Wundheiler (Chicago, Ill.).

Geronimus, Ya. L. Motions with a minimum peak acceleration. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 4, no. 15, 66–91 (1948). (Russian)

The time of motion T and the distance traveled h are given, there is rest at both ends of the run. The author proves first that the peak is a minimum if the acceleration w is $\pm 4h/T^2$ during the first and second half of the motion, respectively. Continuous accelerations arbitrarily close to this peak can be found. Using Chebyshev's method of best approximation, and assuming the motion symmetric about $t=\frac{1}{2}T$ with $w(\frac{1}{2}T)=0$, the author finds complete solutions for 18 combinations of the following conditions in $0\leqq t<\frac{1}{2}T$: (1a) w is a polynomial in t of degree $n\leqq 3$; (1b) $w=A+B\cos\phi+C\cos 2\phi+D\cos 3\phi$, $\phi=\pi t/T$; (2a) $w>0$; (2b) $w>0$, $w'\leqq 0$; (2c) $w>0$, $w'\geqq 0$, $w'(0)=0$; (2d) $w>0$, $w'<0$, $w''\geqq 0$. For polynomials of degree three the condition of given maximum velocity is combined with (2a), (2c), and (2d). The results are neatly tabulated.
A. W. Wundheiler (Chicago, Ill.).

Geronimus, Ya. L. On some problems in the synthesis of cam mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 5, no. 19, 62–81 (1948). (Russian)

The author continues his study of runs with a minimum peak acceleration [see the preceding review]. The distance h, the time T, the maximum speed V_0 are given; the velocity V and acceleration W are continuous and vanish at the beginning and at the end of the run; and the (W, t) curve is symmetric about the point $(0, \frac{1}{2}T)$. A solution of the form $a_1\sin\phi+a_2\sin 2\phi+a_3\sin 3\phi$, $\phi=2\pi t/T$ is determined to yield a minimum over-all acceleration peak, assuming $W(t)\geqq 0$ for $t<\frac{1}{2}T$. Familiar methods are used, and considerable labor expended. Next, the symmetry condition is replaced by prescribing the value of the area under the distance-time curve, and a minimum peak of deceleration demanded. Expressions of the type $a_1\sin\phi+a_2\sin 2\phi$, ϕ linear in t, are used for the acceleration and deceleration periods. As in the first problem, the analysis of some inequalities makes the procedure very laborious.
A. W. Wundheiler (Chicago, Ill.).

Geronimus, Ya. L. Concerning the design of some cam mechanisms with piecewise circular cams. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 5, no. 17, 69–79 (1948). (Russian)

The author continues his work on minimax problems [see the second preceding review]. Cam (A) is translatory of velocity v_1 and two circular arcs separated by a straight line segment. Cam (B) rotates at a speed ω and consists of three circular arcs. For cam (A) the velocity v_1 of the follower is given; for cam (B) the angular velocity ω. In both cases the travel time T and distance h of the follower travel are given. Three types of problems are considered: (1) minimize the absolute peak acceleration w_0; (2) given the ratio w_1/w_2 of the peak acceleration w_1 and peak acceleration w_2, make w_2 a minimum; (3) given the absolute peak velocity v_0, minimize the absolute peak acceleration w_0. The choice of the type of problems is justified by practical considerations. Complete solutions of the six problems involved are presented.
A. W. Wundheiler (Chicago, Ill.).

Žegalov, L. I. A method of geometric loci for the design of the smallest cam for a given plane follower. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 6, no. 21, 69–71 (1949). (Russian)

Twelve positions of the follower determine twelve sides of a polygon approximating the cam, after the center of the cam is chosen on the axis of the follower. As this center is varied, the locus of a vertex of the polygon is a straight line. The loci of the endpoints of the smallest side intersect in a point. The corresponding polygon has (in the limit) the smallest diameter. *A. W. Wundheiler* (Chicago, Ill.).

Kobrinskiĭ, A. E. On kinematic errors of mechanisms in nearly extreme positions. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 9, no. 33, 29–38 (1950). (Russian)

If $\Delta\omega=\sum_s a_s\Delta l_s+R$, where Δl_s are some fundamental errors and R is, at least, of order two in Δl_s, then, near a position where not all a_s are finite, the author vaguely advocates $\max|R|/\sum a_s\Delta l_s$ as a measure of accuracy. The presentation is awkward. *A. W. Wundheiler* (Chicago, Ill.).

Kreines, M. A., and Rozovskiĭ, M. S. A sketch of the angular velocities in a gear box with two degrees of freedom without a fixed external support against rotation. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 2, 5–16 (1947). (Russian)

Transmissions of two degrees of freedom, capable of rigid rotation with the driving shaft, are considered. It is assumed that each transmission ratio i_q is enforced by locking a gear wheel q. In general, its angular velocity (1) $\omega_q=a_q\omega_0+b_q\omega_m+\cdots$, where ω_0 and ω_m are the input and output speeds. In the plane (ω_m, ω) the straight line (1) intercepts on the ω_m-axis the ratio i_q, and goes through the point (ω_0, ω_0). The graph of all the equations (1) is the author's "plan of angular velocities". It can also be used for the determination of the moments on the several wheels.
A. W. Wundheiler (Chicago, Ill.).

Kreines, M. A. The coefficient of efficiency and the transmission ratio of compound gear trains. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 1, 21–48 (1947). (Russian)

The paper deals with gear trains consisting of a number of simple (three-member) epicyclic (planetary) trains, $K_1, \cdots, K_n$ whose arms (i.e., the members connecting the axles) rotate about fixed axes. Only torques are assumed to act at the members, and friction only between the teeth is considered. If i_q is the train ratio (with the arm at rest) of K_q, the over-all transmission ratio is $i=f(i_1, \cdots, i_n)$ where f is a quotient of two linear functions. If η is the efficiency, $\eta=f(i_1\eta_1^{\pm 1}, \cdots, i_n\eta_n^{\pm 1})/f(i_1, \cdots, i_n)$ where the exponent of η_q depends on which of the two wheels of K_q is driving. If all η_q are close to 1, the exponent of η_q is sgn j_q, where $j_q=\partial\log|i|/\partial\log|i_q|$. In this case $\eta\approx 1-\sum|j_q|(1-\eta_q)$. Several numerical examples are given. In general, j_q is the fraction of power lost in K_q. A general method for the determination of f is given in chap. 3. The argument is valid also for spatial gear trains. *A. W. Wundheiler*.

Dobrovol'skiĭ, V. V. On the coefficient of efficiency of compound gear trains. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 1, 59–69 (1947). (Russian)

The results of the preceding paper are stated (independently) in less general terms. *A. W. Wundheiler.*

Dobrovol'skiĭ, V. V. On statically indeterminate mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 5, no. 18, 24–33 (1948). (Russian)

Statically determinate mechanisms are defined here relative to a force system (F) as such for which all reactions are determined when the force system (F) is given, the inertial forces neglected, and the deformations of the rigid members neglected. Examples are given, and also a classification of mechanisms (according to possible translations and rotations of their members) together with a statement of the force systems relative to which the mechanisms are statically determinate. This classification is too cryptic for this reviewer to report on. *A. W. Wundheiler* (Chicago, Ill.).

Dobrovol'skiĭ, V. V. On bevel gearing. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 2, 127–140 (1947). (Russian)

The principles of bevel gearing are reviewed, and octoid and spherical involute teeth considered. For the octoid form the equations of the tooth outlines, and the endpoints of the active section of the line of contact are determined. For the involute form, the minimum number of teeth without undercutting is determined, as well as the coefficient of overlap. There is some discussion of internal meshing and the associated interference phenomena. The method is strictly analytic. Part of the paper must be regarded as expository. *A. W. Wundheiler* (Chicago, Ill.).

Kolčin, N. I. A skew worm gear with arbitrary angle of the axes. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 3, no. 9, 18–51 (1947). (Russian)

The author attaches a Cartesian coordinate system to each element of the gear, and derives the mating surface and the characteristic lines (of contact) for (1) a convolute surface (generated by a straight line in helical motion); (2) its special case, the evolvent surface (the straight line is tangent to the helix described by its central point); and (3) another special case, the Archimedean surface, when the straight line intersects the axis of the helical motion. The developments are given in complete detail. Numerical examples and diagrams of the contact lines are given. *A. W. Wundheiler* (Chicago, Ill.).

Gessen, B. A. The analytical method of investigation of spatial meshing. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 5, no. 19, 22–61 (1948). (Russian)

Chapters 1 and 2 contain a review of the kinematics of a moving surface in parametric treatment. Both vector and coordinate symbolics are used. Envelope equation and relative motion are emphasized. Chapter 3 (5 pp.) rewrites Gohman's equations [Teoriya zaceplenii, Odessa, 1886] of mating surfaces obtained as the two envelopes, relative to the two meshing bodies, of any moving surface. The equations of the characteristics are also given. These equations are of the first order in the partial derivatives of F, where $F(X, Y, Z)=0$ is the equation of the surface in question. In chapter 4 (9 pp.) the equations for the same surfaces and characteristics are derived assuming parametric representation; their left-hand sides are written as determinants. The whole discussion is made in general terms. Most readers would probably be satisfied if they could see the last nine pages only. The developments are applied in the paper of the following review. *A. W. Wundheiler* (Chicago, Ill.).

Gessen, B. A., and Zak, P. S. Globoidal meshing. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 6, no. 21, 27–49 (1949). (Russian)

A globoidal (hourglass) worm gear is considered. The surface of the worm is generated by a straight line tangent to the worm wheel and intersecting the worm axis (perpendicular to the wheel axis) while both revolve uniformly. The equations of both mating surfaces and the lines of contact are developed by the method described in the preceding review. Because of undercutting, only one part of the tooth surface is an envelope of the mating surface. There are two other parts whose equations are also given. It is shown that there is a tooth with which the worm has three simultaneous lines of contact. With other teeth there are two or one lines of contact. Clear diagrams illustrate these facts. *A. W. Wundheiler* (Chicago, Ill.).

Gerasimov, Yu. N. A generalized theory of radial and nonradial Geneva-stop mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 4, no. 15, 20–48 (1948). (Russian)

Geneva-stop mechanisms are generalized by allowing the slots to be nonradial and nonuniformly spaced. Several driving pins become necessary, and special circular exit slots must be provided, the exit occurring during the rest period. Relations are derived between the numbers of slots and pins, and the durations of motion and rest periods. The slot design for shockless operation is discussed. With nonradial slots the shaft separation L can be kept constant while the driven member is varied. The offset of the nonradial slot is determined as a function of L and the number of slots when they are evenly spaced. Formulas are derived for the angular velocity and acceleration of the driven member, and a detailed discussion is made of their variation. Two design examples are given. *A. W. Wundheiler* (Chicago, Ill.).

Kudryavcev, V. N. An investigation of planetary gears. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 3, no. 11, 38–63 (1947). (Russian)

A systematic terminology and classification is given for plane planetary gear trains, but its exhaustiveness is not discussed. The transmission ratios and their variations are considered. Two-thirds of the paper is devoted to a determination of efficiency coefficients, assuming that the arm (=center line of the gear train) is at rest. In fact, "efficiency of planetary trains" would be a more useful title for the paper. *A. W. Wundheiler* (Chicago, Ill.).

Šitikov, B. V., and Ščepetil'nikov, V. A. On the number of satellites in planetary transmissions. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 6, no. 21, 50–68 (1949). (Russian)

Let 1 and 4 be two coaxial gear wheels, and let the compound satellite consist of two solid, keyed, coaxial gears 2 and 3, 3 meshing with the fixed gear 4, and 2 with 3. Then the number of satellites compatible with the gears 1 and 4 is $p=iz_1/mz_3$ where i is the transmission ratio between the arm and the gear 1, z the number of teeth, and m any common divisor of z_2 and z_3. An inverse problem (p and i given) is also discussed, as well as the logic of the assembling procedure. *A. W. Wundheiler* (Chicago, Ill.).

Yudin, V. A. A theory of a planetary, pin-tooth, internal-gear transmission. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 4, no. 13, 42–77 (1948). (Russian)

The outer wheel of an internal-meshing planetary (hypocyclic) gear train is pin geared (the teeth are circular). The pin radius is such that the pitch point is never the point of contact. The profile of the inner wheel is determined as a parallel to a lengthened pericycloid, and two other generations of this profile are given (a parallel to a hypocycloid, and a four-bar curve). The (indeterminate) dynamics of the transmission, and its several losses are discussed. The design problem is assessed. Simplicity of gear production is claimed, as well as advantages for use at high transmission ratios. There is a numerical design example.

A. W. Wundheiler (Chicago, Ill.).

Zinov'ev, V. A. The energy method for the study of machinery motion. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 4, no. 15, 49–65 (1948). (Russian)

The equation of motion of machinery driven by a prime mover (of parameter ϕ), rotating with an angular velocity ω, is written in the form $M_d+M_r=d(\frac{1}{2}I\omega^2)/d\phi$. Eight cases are singled out in terms of the single variable (ϕ, ω or t) on which M_d, M_r or I are assumed to depend. Subcases of integrability are pointed out. *A. W. Wundheiler.*

Federhofer, Karl. Über den Trägheitspol des eben bewegten starren Systems und die Trägheitspolkurve des zentrischen Schubkurbelgetriebes. Österreich. Ing.-Arch. 5, 240–245 (1951).

The inertia pole curve (Trägheitspolkurve) is the locus of the centers of oscillation of a plane body in continuous motion. The locus vector of the center of oscillation, measured from the center of gravity of a plane disc, is completely determined by the center of rotation, the inflection pole (opposite the center of rotation on the inflection circle) and the radius of gyration. The method is applied to the connecting-rod of a crosshead mechanism. In this case, the points of the complete inertia pole curve lie close to a straight line segment carried by the connecting-rod and perpendicular to its axis. *M. Goldberg.*

Bosanac, Eduard. Über den Beweglichkeitsgrad kinematischer Verbindungen. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 6, 57–64 (1951). (Serbo-Croatian. German summary)

Some of the formulas for the number of degrees of freedom of a plane linkage [see, e.g., R. Beyer, Technische Kinematik . . ., Barth, Leipzig, 1931, pp. 22–28] are rederived (with algebraic variations). There are several serious misprints in the paper. [The reviewer observes that the proof of all these (and similar) relations is a matter of routine: one verifies that the equation is preserved after a member is removed.] *A. W. Wundheiler* (Chicago, Ill.).

***Žukovskiĭ, N. E. Teoretičeskaya mehanika. [Theoretical Mechanics].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 811 pp.

This book was prepared from Žukovskiĭ's lectures given at Moscow University during the years 1886–1920. It is divided into three parts of which the largest (602 pp.) is devoted to the mechanics of points and rigid bodies. The other two parts are on hydromechanics (106 pp.) and on the theory of attraction (88 pp.).

Béghin, H. Sur la notion de travail dans la mécanique du continu. Ann. Inst. Fourier Grenoble 2 (1950), 173–184 (1951).

The author remarks that if virtual work in a continuous medium is properly defined, the principle of virtual work has general validity, not limited to frictionless systems or to displacements compatible with the constraints.

C. A. Truesdell (Bloomington, Ind.).

de Possel, René. La notion physique d'énergie vis-à-vis des définitions du travail et de la force. Ann. Inst. Fourier Grenoble 2 (1950), 185–195 (1951).

The author considers the formulation of classical mechanics given by Brelot [Les principes mathématiques de la mécanique classique, Arthaud, Grenoble-Paris, 1945; these Rev. 7, 223], in which matter may be in part continuous and in part discrete, so that the apparatus of Lebesgue-Stieltjes integrals etc. can be employed. He gives a definition of power and shows that it is possible to formulate a Hertzian mechanics in which energy is primitive and force a derived concept.

C. Truesdell (Bloomington, Ind.).

Nadile, Antonio. Sull'esistenza per i sistemi anolonomi soggetti a vincoli reonomi di un integrale analogo a quello dell'energia. Boll. Un. Mat. Ital. (3) 5, 297–301 (1950).

A convenient form is given to the equations of non-holonomic dynamical systems with rheonomic constraints by the introduction of an extra Lagrangian coordinate and it is then shown how in some cases the system admits a first integral analogous to an energy integral. *D. C. Lewis.*

Bottema, O., and Beth, H. J. E. The stationary motions of a rigid body under no forces in four-dimensional space. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 123–129 (1951).

The authors summarize their results as follows: "If a rigid body in four-dimensional space is moving under no forces about a fixed point, the only stable stationary motions are orthogonal double rotations about two skew planes of dynamical symmetry, namely: (1) about the two planes of which one contains the two largest axes of inertia; (2) about the two planes of which one contains the largest and the smallest axis of inertia." *D. C. Lewis* (Baltimore, Md.).

Rubašov, A. N. The motion of the principal axes of inertia in a body of variable mass. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 385–386 (1951). (Russian)

A formula for the angular velocity of rotation of the principal axes of inertia in a body of variable mass is given:

$$\omega=\int\{[yz/(C-B)]\mathbf{i}+[zx/(A-C)]\mathbf{j}+[xy/(B-A)]\mathbf{k}\}\frac{\partial\rho}{\partial t}d\tau.$$

E. Leimanis (Vancouver, B. C.).

Zeuli, Tino. Generalizzazione del metodo di Newcomb per lo studio delle vibrazioni pseudo armoniche. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 84, 31–40 (1950).

In order to solve the equation of motion of a vibrating point

$$\ddot{q}+\sigma^2q+\sum_{n=2}^{\infty}a_nq^n=0,\quad (q)_0=a,\quad (\dot{q})_0=b,$$

the author introduces a new parameter $\tau=\sigma t(1+\mu)^{-\frac{1}{2}}$ and imposes on the constant μ the condition that the solution of the new equation be of period 2π. Both quantities μ and q are considered as power series in a and b, of which a few terms are computed. *J. Haantjes* (Leiden).

Hydrodynamics, Aerodynamics, Acoustics

√★**Neményi, P. F. Recent developments in inverse and semi-inverse methods in the mechanics of continua.** Advances in Applied Mechanics, vol. 2, edited by Richard von Mises and Theodore von Kármán, pp. 123–151. Academic Press, Inc., New York, N. Y., 1951. $6.50.

The author calls inverse "an investigation of a partial differential equation of physics if in it the boundary conditions (or certain other supplementary conditions) are not prescribed at the outset. Instead, the solution is defined by the differential equation and certain additional analytical, geometrical, kinematical, or physical properties of the field. In the semi-inverse method some of the boundary conditions are prescribed at the outset, whereas others are left open and obtained indirectly". This article does not attempt to present the subject exhaustively, but aims rather "to elucidate the nature, value, and the potentialities of this approach". Incidentally, however, it explains the essence of nearly all the main contributions.

The paper is divided into sections of about equal length concerning inviscid incompressible fluids, perfect gases, elastic bodies in equilibrium, and plastic bodies. Much of the work mentioned is very recent, some unpublished. The author in many cases is able to correlate, compare, and contrast the various investigations, often in different fields. At the end of the paper is a table summarizing results known up to the present on five semi-inverse problems in seven domains of continuum mechanics. Among the more interesting investigations cited are those of Taylor and Trkal on decaying motion of a viscous fluid; those of Tollmien, Prim, and Neményi-Prim on limiting lines, problems of invariance, and "generalized Beltrami flows" in gas dynamics; and those of Neményi on an "influence principle" and on stress trajectories in elasticity.

The author emphasizes the value of simple exact solutions as typical and suggestive illustrative cases, and he shows that such solutions have been or can be obtained by inverse or semi-inverse methods in many cases. His examples, too numerous to list here, substantiate also the following conclusions. "1. Inverse and semi-inverse methods [may] lead to solutions of important boundary value problems." "2. [They] may lead to the discovery of unsuspected discontinuities, limitations, or general field properties . . ." "3. [They] may settle existence questions in a positive sense, or may decide a uniqueness problem in a negative sense." "4. [They] are essential for the comparative study of the differential equations of the various problems of mechanics."

It is unfortunate that so valuable and stimulating a work has been so carelessly printed. There is an abundance of misprints and grammatical errors, and for the running heads the Academic Press has invented the new word "Continua mechanics." *C. Truesdell* (Bloomington, Ind.).

Parsons, D. H. Fluid motions whose kinematics are independent of the compressibility of the fluid. Quart. J. Mech. Appl. Math. 3, 446–451 (1950).

The author proves that if a steady flow of an inviscid incompressible fluid and of an inviscid compressible fluid (with barotropic equation of state, $p=p(\rho)$) have the same velocity field, then the velocity magnitude must be constant on streamlines. [It is the reviewer's opinion that the conclusion is still true if the two flows have only their streamlines in common. For results in this direction, see the recent article by Neményi [see the preceding review].] The author then shows that if the flows are specialized to the plane they must have a streamline pattern of either concentric circles or parallel lines. This is a special case of a more general result by Neményi and Prim [J. Math. Physics 27, 130–135 (1948); these Rev. 10, 73]. For the axially symmetric case, mentioned but not treated by the author, Prim [ibid. 28, 50–53 (1949); these Rev. 10, 634] has shown that the flows with constant velocity magnitude on streamlines are purely axial. *D. Gilbarg* (Bloomington, Ind.).

Ballabh, Ram. On coincidence of vortex and stream lines in ideal liquids. Ganita 1, 1–4 (1950).

In the flow of a homogeneous incompressible fluid let (∗) $E=\lambda q$, where E is the vorticity vector, q is the velocity vector, and λ is a scalar function of coordinates and time t. The author shows (1) if the fluid is viscous and $\lambda=\lambda(t)$, then λ is an absolute constant, (2) if the fluid is ideal, then λ is independent of t, and the flow is steady. The class of flows in which (∗) holds, $E\neq 0$, was first studied by Gromeka [Some properties of flows of incompressible fluids, Thesis, Kazan, 1881], and Beltrami [Opera, vol. 4, Hoepli, Milano, 1920, pp. 300–309], with whose name these flows are usually associated. As a partial survey of the known results on the subject may be mentioned G. Popov [Vestnik Moskov. Univ. 3, no. 8, 35–47 (1948); these Rev. 11, 269].
D. Gilbarg (Bloomington, Ind.).

Ballabh, Ram. Coincidence of vortex and stream lines in a liquid of variable density. Ganita 1, 39–43 (1950).

The author here considers Beltrami flows [see preceding review] in an incompressible viscous fluid in which the density varies with depth only. He finds from the continuity equation the general form of the solution in case the flow velocity is constant in each horizontal plane at any time, and shows, using the equations of motion, that there is no steady motion of this kind under gravity. *D. Gilbarg.*

Müller, W. Über den Impulssatz für einer in der Flüssigkeit bewegten Körper. Ing.-Arch. 18, 338–343 (1950).

Vector expressions, in terms of the impulse, for the force and moment on a solid moving through liquid are obtained in the same form as in the reviewer's Theoretical Hydrodynamics [Macmillan, New York, 1950, chap. 17; these Rev. 11, 471]. Expressions are given for virtual mass and moment of inertia in the case of an oblate spheroid.
L. M. Milne-Thomson (Greenwich).

Vladimirsky, Serge. Théorie du mouvement non stationnaire d'une plaque mince par la méthode du potentiel. C. R. Acad. Sci. Paris 233, 352–354 (1951).

Biésel, F. Remarques sur la célérité de la houle irrotationnelle exacte au troisième ordre. Houille Blanche 6, 414–416 (1951).

In the theory of waves of finite amplitude in water of finite depth let the motion be converted to steady motion by imposing a velocity c opposite to the direction of propagation. There is then a movement with velocity c' of the center of gravity of the fluid contained between two vertical planes a wave length apart. The author computes c', retaining terms of the third order, in terms of the mean depth, the wave length and amplitude. *J. V. Wehausen.*

Spraglin, William E. Flow through cascades in tandem. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2393**, 44 pp. (1951).

The problem of finding the blade shapes for the incompressible, inviscid, two-dimensional flow with prescribed turning and surface velocity distributions through cascades in tandem is treated by conformal mapping techniques and a method of solving the direct problem is indicated. The canonical flow in the region bounded by concentric circles is discussed in detail. *C. Saltzer* (Cleveland, Ohio).

Mattioli, Gian Domenico. Calcolo del campo di velocità indotta per l'ala rettangolare. Aerotecnica **24**, 281–287 (1949).

In a previous publication [Ing.-Arch. **10**, 153–159 (1939)] the author has written down the integro-differential equation for the flow field about a thin wing moving, with a constant speed U and angle of attack, through an incompressible inviscid fluid. If (x, y) are the chordwise and spanwise coordinates, (ξ, η) the corresponding coordinates of integration, and μ the local doublet strength, then

$$\int\int \frac{[(\xi-x)^2+(\eta-y)^2]^{\frac{1}{2}}}{(\xi-x)(\eta-y)} \frac{\partial^2\mu}{\partial\xi\partial\eta} d\xi d\eta = \int_{-h}^{h} \left(\frac{\partial\mu}{\partial\eta}\right)_{\xi=c(\eta)} \frac{d\eta}{(\eta-y)} + U \sin \alpha.$$

Here the double integral extends over the wing surface giving the local contribution of the doublet distribution. The single integral extends over the trailing edge $\xi=c(\eta)$ and essentially serves to satisfy the Kutta condition at the trailing edge.

In the present paper, the author sets about to compute the flow field generated by a rectangular wing of semispan h and constant chord c. This is carried out by an approximate solution of the above equation, the approximation being involved with the kernel $[(\xi-x)^2+(\eta-y)^2]^{\frac{1}{2}}/(\xi-x)(\eta-y)$ and the form of the doublet distribution $\mu(\xi, \eta)$. To carry out these approximations the author divides the space of the independent variables (x, y) into two regions on the wing, Σ_m: $|y| \leqq h-1$ and Σ_e comprising the two remaining rectangles $h \leqq |y| \leqq h-1$. For the integration variables (ξ, η) he differentiates between the regions $\Sigma_{\rm I}$: $|\eta-y| \leqq 1$ and $\Sigma_{\rm II}$: $-h \leqq \eta \leqq y-1$; $h \geqq \eta \geqq y+1$. Now for each combination of $\{\Sigma_m, \Sigma_e\}$ with $\{\Sigma_{\rm I}, \Sigma_{\rm II}\}$ suitable approximations for the kernel and form of doublet distribution have been found so that the actual doublet distribution corresponding to the rectangular wing may be evaluated. The author tabulates all of the (23) integrals involved as chordwise series expansions, including cubic terms. Singular values of the derivatives of the integrals are given for the leading and trailing edges. *F. E. Marble* (Pasadena, Calif.).

van Heemert, A. The calculation of downwash fields for a lifting plane in steady flow. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 51**, i+42 pp. (1949).

For a thin wing of arbitrary planform in steady incompressible flow, the problem of determining the double-layer distribution to fit the specified downwash distribution is considered. A plane trailing vortex wake is assumed and the Kutta-Joukowski trailing-edge flow condition is imposed. Two approximate methods are developed for numerical calculation. One consists of dividing the planform into trapezoidal parts, approximating to the doublet distributions over these parts in a convenient form, and calculating the corresponding downwash due to each part. In the other method, the same sort of assumption is made regarding the doublet distribution, but the numerical integration is carried out somewhat differently. The scheme of practical application of the second method is worked out in detail. No actual results are yet available. The general idea of these methods, as well as the form assumed for the doublet distribution functions, are equivalent to what have been used by a number of authors such as Blenk [Z. Angew. Math. Mech. **5**, 36–47 (1925)] and Weissinger [Math. Nachr. **2**, 45–106 (1949); these Rev. **11**, 64]. The author believes that the new treatment offers advantages of accuracy and mathematical rigor. *W. R. Sears* (Ithaca, N. Y.).

Greidanus, J. H., and van Heemert, A. Chordwise downwash distribution of an infinite wing of constant chord with a periodic spanwise distribution of vorticity, in oblique flow. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 78**, i+20+4 pp. (1950).

Here the equations of lifting-surface theory are specialized for the case described in the title, and are simplified in two ways: (1) the angle of yaw is assumed small, and (2) the radial distance from a doublet element to a point of the wing is put equal to the difference of their spanwise (oblique) coordinates, except when this leads to a divergent integral. For straight, unyawed wings, for example, this second approximation, consistently applied, leads to Prandtl's lifting-line equations. The results of this approximate calculation, for a case of 30° yaw, are compared with those of the more complete lifting-surface theory worked out in the paper reviewed above; agreement is good. Finally, a similar comparison is made with results of a generalized Prandtl theory by one of the authors (van Heemert), as yet unpublished; again the agreement is considered good, for wave lengths down to about two chords. *W. R. Sears* (Ithaca, N. Y.).

van Heemert, A. The calculation of the downwash of swept back wings in the region of the tail unit. National Luchtvaartlaboratorium, Amsterdam. Report **F. 73**, i+17 pp. (1950).

This is a calculation for incompressible steady flow, with a subsequent correction of the Prandtl-Glauert type for compressibility. The wing is assumed to have straight, sweptback quarter-chord lines, and is accordingly replaced by a vee-shaped lifting line. The downwash and its vertical and lateral derivatives are computed for points on the centerline directly behind the point of the vee. Displacement of the trailing vortex sheet is neglected, and also the plane of the vee is assumed to include the stream velocity vector. No restrictive assumptions are made about the circulation distribution or its symmetry. *W. R. Sears.*

Jones, E. E. The effect of the non-uniformity of the stream on the aerodynamic characteristics of a moving aerofoil. Quart. J. Mech. Appl. Math. **4**, 64–77 (1951).

The work of Morris [Proc. Roy. Soc. London Ser. A. **161**, 406–419 (1937); **164**, 346–368 (1938); **172**, 213–230 (1939); **188**, 439–463 (1947); these Rev. **1**, 90; **8**, 542] concerning the nonsteady motion of cylindrical airfoils in two-dimensional inviscid incompressible flow is here extended to the case where the cylinder moves in a non-uniform curved stream. For this investigation the complex potential of the undisturbed stream is written as $\sum_{n=1}^{4} A_n z^n$ where z is the complex coordinate in the plane of the flow; thus the results are expected to give only the effects of local stream curvature near the cylinder. The additional terms thus introduced

into the potential, the total force, and the moment on the airfoil of Morris' paper are worked out. The existence of a trailing wake of vortices is considered. As an example, a thin Joukowski airfoil without trailing wake is treated in detail. *W. R. Sears* (Ithaca, N. Y.).

Preston, J. H. Non-steady flows under asymptotic suction conditions. Quart. J. Mech. Appl. Math. 3, 435–445 (1950).

The transient flow due to the alteration of the speed of rotation of a cylinder in a fluid under conditions of suction through the surface is mathematically investigated. Graphs showing its development with time are presented; and it is shown that the circulation round a large circuit embracing the cylinder does not change. [Reviewer's note: The author's conclusions from this theory concerning the effect of withdrawing a Thwaites flap are not borne out by experiment; the resulting flow is found to be highly unstable.]
M. J. Lighthill (Manchester).

Goland, Leonard. A theoretical investigation of heat transfer in the laminar flow regions of airfoils. J. Aeronaut. Sci. 17, 436–440 (1950).

This paper develops a method for calculating approximately the laminar heat transfer from an airfoil with arbitrary chordwise distribution of pressure, based upon the observation that, for constant surface temperature, the temperature distribution in the thermal boundary layer is similar to the velocity distribution in the spanwise boundary layer when the airfoil is yawed. The distinct advantage of this procedure is, of course, that the solution for spanwise boundary layer of a yawed infinite cylinder has been given by Sears [same J. 15, 49–52 (1948); these Rev. 9, 476] and therefore the author need only adopt it to his problem. If x and z are coordinates chordwise and normal to the surface, respectively, and (u, w) the corresponding velocity components, the boundary layer equation for the airfoil surface is $u\partial u/\partial x+w\partial u/\partial z=-\rho^{-1}\partial p/\partial x+\nu\partial^2u/\partial z^2$, where p and ρ are the local pressure and constant density, and ν is the kinematic viscosity of the fluid. The corresponding energy equation for the temperature T is

$$u\partial T/\partial x+w\partial T/\partial z=(\kappa/c_p\rho)\partial^2T/\partial z^2.$$

The presence of pressure gradient in the former equation invalidates any similarity of the velocity and temperature profiles. However if v is the spanwise velocity along the corresponding yawed wing, the spanwise boundary layer satisfies the equation $u\partial v/\partial x+w\partial v/\partial z=\nu\partial^2v/\partial z^2$ where the pressure derivative is absent because the pressure is constant along the span. For a constant surface temperature and Prandtl number of unity ($c_p\mu/\kappa=1$), the mathematical problems for the temperature and spanwise velocity distributions are identical. Then the aforementioned solution of Sears for the transverse boundary layer may be used as it stands.

Calculations of the heat transfer coefficient are given for an example of a symmetrical airfoil when the potential velocity distribution is of the form $u_1\sim(\xi-\xi^3)$; $\xi=x/L$ where L is the airfoil chord. The results are found superior to other approximate solutions which neglect or incompletely account for the lack of similarity between the temperature boundary layer and the chordwise velocity boundary layer. The comparison with the results of Squire are quite favorable. Although it does not seem to be explicitly stated in the paper, it should be mentioned that the stipulation of incompressibility must be taken seriously; that is, the results are valid for liquids which may be considered strictly incompressible and for which the use of c_p or c_v is indifferent. The results are also valid for a gas when the heat transfer rate (and hence the volume dilatation) is infinitesimal, if c_p is replaced by c_v in the author's work. That is, the equation governing the temperature distribution should read

$$u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}=\frac{\kappa}{\rho c_v}\left(\frac{\partial^2T}{\partial z^2}\right)$$

instead of that given in the paper [see Goldstein, Modern Developments in Fluid Dynamics, vol. II, Oxford, 1938, pp. 606–607, eq. (13) and (16)]. For a gas, this change would modify the results slightly. *F. E. Marble.*

Fabri, Jean, et Siestrunck, Raymond. Sur les machines axiales périodiques. C. R. Acad. Sci. Paris 231, 115–117 (1950).

This brief communication discusses the flow of an ideal incompressible fluid through an axial turbomachine consisting of an infinite number of identical stages, each composed of the usual stationary and rotating members. The blade shape is "prescribed" by specifying essentially the flow angle at each blade row. Now it is known that if a solution for the flow exists in such a case, it must be periodic and the authors observe that there should be no difficulty in calculating the flow using the linearized solution proposed by the reviewer [J. Aeronaut. Sci. 15, 473–485 (1948); these Rev. 10, 164]. The authors suggest further that the process of establishing this periodic flow may be calculated employing what is usually known as the "simple radial equilibrium" flow. A pair of difference equations is given for the successive calculation of the flow development. The authors present no example of their suggested program. There has been some experience in the past indicating that the reward of this type of calculation may not be commensurate with the laborious computation involved. *F. E. Marble.*

Siestrunck, R., and Fabri, J. Écoulements tourbillonnaires dans les machines axiales. O.N.E.R.A. Publ. no. 45, ii+64 pp. (1950).

In this paper axial turbomachines are studied under the assumption of an infinite number of blades. Thus the stream function $\psi(r, z)$ involves only two of the cylindrical coordinates r, θ, z and the radial and axial velocity-components may be written as $u=(\rho_0/r\rho)\partial\psi/\partial z$ and $w=-(\rho_0/r\rho)\partial\psi/\partial r$, respectively. A second function γ is introduced for the tangential velocity-component by the definition $\gamma=rv$, and shown to be a function just of ψ. It is found that a non-linear partial differential equation of second order connects ψ and γ with E_a, the total absolute enthalpy (also a function just of ψ). When the flow is incompressible, the equation reduces to $D=r^2E_a'-\gamma\gamma'$, where $D=(\partial/\partial r)^2-(1/r)(\partial/\partial r)+(\partial/\partial z)^2$. Conditions for the flow to be potential either completely or just in the meridian plane, are developed; each wheel, however, is regarded as a vortex disc across which finite jumps occur in γ and in certain derivatives of ψ. Several methods are used to linearize the differential equation, in the case of both incompressible and compressible flows. Various expansions for ψ are then obtained in terms of asymptotic solutions at $z=\pm\infty$ and in terms of solutions taking on specified jumps across the wheels. Both single-wheel and multi-stage turbomachines are investigated. *M. Marden.*

Traupel, W. Kompressible Strömung durch Turbinen. Schweiz. Arch. Angew. Wiss. Tech. 16, 129–138, 176–186 (1950).

The author extends, in this paper, his previous work on the theory of axial turbomachines [e.g., Neue allgemeine Theorie der mehrstufigen axialen Turbomaschinen, Leemann, Zürich, 1942] to the flow of an ideal compressible fluid through a turbine with an infinite number of blades in each blade row. The exact, or even approximate, solution to this problem is quite involved and the author simplifies the situation by either neglecting, or making assumptions concerning, the radial acceleration terms in the equations of motion. The solution given corresponds to that generally known as the simple radial equilibrium solution in which the centrifugal force of the fluid element moving about the axis is balanced only by the radial pressure gradient. The effect of the streamline curvature in the meridianal plane is neglected. For an incompressible fluid the problem reduces to one of an ordinary differential equation, but for the compressible fluid the problem involves at least a set of ordinary differential equations which must usually be solved graphically.

The flow between coaxial cylinders is first investigated when the imposed tangential velocity is inversely proportional to the radius. The radial distribution of mass flow as well as the total mass flowing through the annulus is given. The more general flow between coaxial cylinders is treated; in particular, the important case where the product of mass density and axial velocity is constant over the radius is calculated and the velocity distribution given. Retaining his assumption of constant mass flow distribution over the radius, the author calculates the flow through a complete turbine stage consisting of a nozzle ring (stationary blade row) and the rotating blade row. More important in the theory of turbines is the compressible flow through a turbine wheel in a divergent passage. Consequently the author considers, finally, the corresponding problem where the inner and outer walls consist of coaxial cones. Here the author finds it necessary to assume that all stream surfaces are conical and therefore the geometry of the flow is given by two angles: The direction in the meridianal plane and an angle measured with respect to this plane. The author then calculates the details of the absolute and relative flow angles, as well as the Mach number distribution. *F. E. Marble.*

Hawthorne, W. R. Secondary circulation in fluid flow. Proc. Roy. Soc. London. Ser. A. 206, 374–387 (1951).

The author's summary is as follows: Secondary circulation appears after a fluid with a non-uniform velocity distribution passes round a bend. It alters the character of the flow and is a source of loss. A general expression is developed for its change along a streamline in a perfect, incompressible fluid. The flow in bent circular pipes is analyzed and the theory is compared with experiments on bent pipes and rectangular ducts. In bends the secondary flow is not spiral but oscillatory, the direction of the circulation changing periodically. The theory shows that secondary circulation remains unchanged if streamlines are geodesics on surfaces of constant total pressure. *W. J. Nemerever.*

Viguier, Gabriel. Circulation d'un fluide visqueux incompressible. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 397–405 (1951).

The author calculates the material rate of change of circulation about a fluid circuit in an incompressible fluid where the stress is of the form proposed by M. Girault [Publ. Sci. Tech. Ministère de l'Air, Paris, no. 4 (1931), see ch. III], in which terms of the second order in the rate of deformation are annulled, but terms of third order are retained. He does not interpret his general result, being content to draw only the conclusion that in two very special cases the circulation is conserved. [Reviewer's note. The incompleteness of Girault's theory has been noted by the reviewer [J. Math. Pures Appl. (9) 29, 215–244 (1950), footnote 27; these Rev. 12, 761]. While the author states that he generalizes a result of I. Carstoiu [C. R. Acad. Sci. Paris 224, 534–535 (1947); these Rev. 8, 540], in fact the corresponding result for the classical theory was given by V. Bjerknes [Meteorol. Z. 19, 97–108 (1902)], G. Jaffé [Phys. Z. 21, 541–543 (1920)], and in greater generality by H. Jeffreys [Proc. Cambridge Philos. Soc. 24, 477–479 (1928)]; cf. C. Truesdell [C. R. Acad. Sci. Paris 227, 821–823 (1948); these Rev. 11, 221; J. Meteorology 6, 61–62 (1949)]. The corresponding local vorticity formula for an arbitrary continuous medium was derived and interpreted by the reviewer [C. R. Acad. Sci. Paris 227, 757–759 (1948); these Rev. 10, 490]. Finally, for the two special motions considered by the author it is trivial to remark that the curl of the acceleration is zero, and hence by a theorem of Hankel [Zur allgemeinen Theorie der Bewegung der Flüssigkeiten, Göttingen, 1861, see §8] and Kelvin [Mathematical and Physical Papers, vol. 4, Cambridge Univ. Press, 1910, pp. 13–16; see §59(e)] that these motions if dynamically possible at all are necessarily circulation-preserving in any continuous medium whatever.] *C. Truesdell* (Bloomington, Ind.).

***Schlichting, Hermann. Grenzschicht-Theorie.** Verlag und Druck G. Braun, Karlsruhe, 1951. xv+483 pp. $10.00; bound $10.50.

Here one of the well-known contributors to the theory of boundary layers presents a comprehensive review of the subject as of about 1949 or 1950. It is encyclopaedic in its coverage of the subject and should be of considerable importance to students and research workers. Actually, the subject of viscous flow, not only in boundary layers but in general, is pretty well covered here. The main part of the book is devoted to laminar boundary layers, including the compressible case, heat transfer, unsteady flow, and such up-to-date matters as three-dimensional flow, control by suction, and shock-wave interaction. Laminar-boundary-layer instability is discussed in detail, with attention to experimental results, in a chapter on transition.

The turbulent case is not as completely treated, as might be expected. The mixing-length theories are given, and the Buri and Gruschwitz semi-empirical methods. The statistical theories of turbulence, including the Weizsäcker-Heisenberg hypothesis, are only mentioned. There is nothing included on heat transfer through the turbulent layer, a fact which seems to the reviewer difficult to understand or to excuse. One would expect that the theories of Taylor, Prandtl, and von Kármán [reported by von Kármán, Trans. A.S.M.E. 61, 705–710 (1939)] would be included, as well as more recent Russian work and some of the numerous experimental results.

The reader may notice some other omissions, some due, no doubt, to failure of American and English research results to reach the author in Germany. In general, however, the attention to the published literature is excellent. In fact, the reader will be impressed by the great size to which the literature on boundary layers has grown since Prandtl's

1904 paper. Unfortunately, the subject index is far too brief for a book of this sort, but it is augmented by a complete author index. *W. R. Sears* (Ithaca, N. Y.).

***Berker, Ratip. Sur une solution des équations de la couche limite.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 83–96.

Consider the boundary layer equation for an incompressible fluid without pressure gradient

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2},\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

where x, y are the rectangular coordinates parallel and normal to the direction of flow, (u, v) the corresponding velocities, and ν the kinematic viscosity. Choosing the usual stream function ψ: $u=\partial\psi/\partial y$, $v=-\partial\psi/\partial x$, the author produces a solution of these equations by assuming ψ in the form $\psi=\psi\{yf(x)+g(x)\}$, $f(x)$ and $g(x)$ being unknown functions. Calling $w=C[y/A(x_0-x)+g(x)]$ where A, C, and x_0 are constants, his solution may be written $\psi=C^{-1}\int\wp(w;0,g_3)dw$; here $\wp$ is a Weierstrass elliptic function and g_3 is a constant depending upon the other constants of the system. The function $g(x)$ and the three constants are available for satisfying the boundary conditions, but unfortunately the author does not seem able to find a physical problem whose natural boundary conditions may be satisfied exactly. It is, however, possible to approximate the solution for the boundary layer on a semi-infinite flat plate without pressure gradient, the approximation arising in the attempt to attain the free-stream velocity as the edge of the boundary layer is approached. The solution is good only very far from the leading edge of the plate where the author states that his results check those of Blasius within 1 percent near the plate and 5 percent near the outer edge of the boundary layer.

Some of the results seem, in the reviewer's opinion, to make this solution quite inappropriate for a solution of Blasius' problem. For example, the boundary layer streamlines emerge from the solution as straight lines originating from a common point ahead of the plate, and the edge of the boundary layer turns out to be a streamline. It seems quite certain that the solution produced by the author is either reducible to, or is closely related to, the solutions of Hamel [Jber. Deutsch. Math. Verein. **25**, 34–60, 60–65 (1916)], Jeffery [Philos. Mag. (6) **29**, 445–455, 455–465 (1915)] and von Kármán [Vorträge aus dem Gebiete der Hydro- und Aerodynamik (Innsbruck, 1922), Springer, Berlin, 1924] which are valid for a divergent channel. *F. E. Marble.*

Timman, R. A calculation method for threedimensional laminar boundary layers. II. The potential flow about a yawed ellipsoid at zero incidence. Nationaal Luchtvaartlaboratorium, Amsterdam. Report F. 74, i+16 pp. (3 plates) (1950).

In a previous report the author gave a method for calculation of three-dimensional laminar boundary layers over smooth surfaces. Contemplating the application of the method to an ellipsoid of three unequal axes, the author here calculates the surface streamlines of incompressible potential flow about such a body. The formulae are taken from standard reference works [H. Lamb, Hydrodynamics, Cambridge Univ. Press, 1932; W. F. Durand (ed.), Aerodynamic Theory, vol. I (article by M. Munk), Springer, Berlin, 1934]. Numerical results are tabulated and plotted. The special case considered has axes of lengths 3, 1, and 0.15 in the x, y, and z directions. The incidence is zero and the yaw 45°; i.e., the stream direction bisects the angle between the x and y directions. *W. R. Sears.*

Cole, Julian D. On a quasi-linear parabolic equation occurring in aerodynamics. Quart. Appl. Math. **9**, 225–236 (1951).

L'équation non linéaire du type elliptique:

$$(1)\qquad \partial u/\partial t+u\partial u/\partial x=\nu\partial^2u/\partial x^2$$

intervient dans diverses questions d'aérodynamique, en particulier dans la théorie approchée de l'onde longitudinale monodimensionnelle avec viscosité [cf. Lagerstrom, Cole, and Trilling, Problems in the theory of viscous compressible fluids, California Institute of Technology, 1949; ces Rev. **12**, 873], et dans un schéma mathématique de la turbulence donné par Burgers [Advances in Applied Mechanics, vol. 1, pp. 171–199, Academic Press, N. Y., 1948; ces Rev. **10**, 270].

Le travail est consacré à l'étude de l'équation (1) en relation avec les applications aérodynamiques. L'auteur forme les solutions de (1), définies pour $t\geqq 0$ par la donnée des valeurs de u à l'instant initial: (2) $u(x,0)=u_0(x)$. La considération de l'équation de la chaleur: (3) $\partial\theta/\partial t=\nu\partial^2\theta/\partial x^2$ permet d'arriver facilement à ce résultat. Si θ est une solution générale de (3), alors (4) $u(x,t)=-2\nu\theta_x/\theta$ est une solution générale de (1) et la condition (2) se traduit pour θ par une condition de même forme: (5) $\theta(x,0)=\theta_0(x)$, où $\theta_0(x)$ est définie en fonction de $u_0(x)$. Comme on sait expliciter pour des conditions limites de ce type les solutions de l'équation de la chaleur, on peut expliciter la solution du problème envisagé.

L'auteur utilise ce résultat pour étudier l'évolution d'une onde de choc dans un milieu visqueux. Retenons que l'effet de la viscosité est de supprimer la discontinuité initiale des vitesses. Le deuxième exemple est celui où $u_0(x)$ est une fonction périodique, ce qui permet de mettre en évidence l'amortissement d'une perturbation initiale périodique ou encore l'amortissement de la turbulence dans une enceinte. L'allure des solutions en fonction du temps est étudiée en détail dans le cas particulier d'une perturbation sinsuoïdale. On voit en particulier que si le nombre de Reynolds R est petit, il devient légitime de négliger les termes non linéaires de l'équation étudiée, tandis que l'allure des solutions change considérablement quand R croît. *R. Gerber* (Grenoble).

Krzywoblocki, M. Z. On complete forms in a turbulent three-dimensional flow of compressible viscous fluid. Österreich. Ing.-Arch. **5**, 129–137 (1951).

Prandtl has suggested a generalized tensor form for Reynolds stresses in an incompressible turbulent flow in cases where the momentum transfer theory is applied and where neither the mean nor the turbulent motions are confined to two dimensions. In the present note there is suggested a sequence of generalized tensor and vector forms for additional stresses and energy terms in a compressible viscous turbulent three-dimensional flow. Four basic equations are taken into account: equation of motion, continuity, state and energy. As the basic theory, the mixture length theory was assumed and in the equation of motion the momentum transfer theory was accepted. *Author's summary.*

Krzywoblocki, M. Z. On the asymptotic expansion in three-dimensional compressible viscous flow. J. Franklin Inst. **250**, 213–217 (1950).

Goldstein's laminar wake theory [Proc. Roy. Soc. London. Ser. A. **142**, 545–562 (1933)] is extended to allow for compressibility. The analysis is purely formal.

M. J. Lighthill (Manchester).

Howarth, L. Some aspects of Rayleigh's problem for a compressible fluid. Quart. J. Mech. Appl. Math. **4**, 157–169 (1951).

C'est l'extension à un fluide compressible du problème de Rayleigh dans lequel un plan indéfini limitant le fluide au repos prend instantanément une vitesse tangentielle constante. Une solution approchée est donnée pour la variation de la pression avec le temps au contact du plan, dans le cas particulier où le nombre de Mach est petit et où le nombre de Prandtl est pris égal à $\frac{3}{4}$ (valeur pour l'air: 0.70). Indiquons les résultats numériques suivants: l'accroissement instantané de la pression est égal à $0.221M^2p_s$, où M est le nombre de Mach et p_s la pression au repos; cet excédent de pression ne vaut plus que la moitié de sa valeur initiale au bout de 5×10^{-10} secondes et le dixième après 10^{-8} secondes.

L'auteur souligne que pour des temps de cet ordre les phénomènes moléculaires ne permettent plus d'appliquer correctement les équations des milieux continus. Mais on peut tout de même conclure du calcul que pour des temps de l'ordre de 10^{-4} secondes, pour lesquels les équations demeurent valables, la pression a pratiquement repris sa valeur initiale. Enfin les équations complètes du mouvement commençant sont étudiées dans le cas simplifié où le coefficient de viscosité est constant. *R. Gerber* (Grenoble).

✓ ∗**Goldstein, Sydney. Linearized theory of supersonic flow.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1950. ii+22 pp.

In this lecture series, topics such as some general theorems in linearized theory, conical flows, the methods of sources and sinks and so on are briefly but precisely stated. The latter half of the lectures is devoted to the techniques of improvement of the linearized approximation, the solution near a singular characteristic and solution at infinity. Simple examples are cited in these discussions. *Y. H. Kuo.*

∗**Handbook of Supersonic Aerodynamics.** U. S. Navy, Bureau of Ordnance, NavOrd Rep. 1488, Washington, D. C., 1950. Vol. I, iv+400 pp. (1 plate). $1.75; Vol. II, iii+197 pp. (not consecutively paged). $1.50 (for sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C.)

The volumes consist principally of tables, with the minimum of explanatory text, on the properties of the atmosphere up to about 500,000 feet as supposed at present, on steady one-dimensional flow problems, flow at constant entropy and across oblique shocks. The data on normal shocks includes the cases of detonation and condensation waves. *M. J. Lighthill* (Manchester).

Giese, J. H. Compressible flows with degenerate hodographs. Quart. Appl. Math. **9**, 237–246 (1951).

In this paper, the author considers the three-dimensional flow of a perfect, compressible, irrotational fluid with degenerate hodograph and discusses the geometric relations between the physical flow and its hodograph map. A brief statement of the basic mapping theorem is that the map in the physical space of a point P of a one (two) dimensional hodograph is contained in a plane (line) normal to the hodograph map at P. It is shown that simple waves consist of arcs of conically deformed Prandtl-Meyer epicycloids and conversely. To determine the properties of double waves (two-dimensional hodographs), the author introduces the first and second fundamental tensors of the hodograph surface and studies various properties of the characteristic curves. The relation of a particular class of double waves and the flows of Taylor-Maccoll and Busemann are discussed. Finally, the author considers the problem of constructing flows (in the small) with axisymmetric hodographs.

N. Coburn (Ann Arbor, Mich.).

Tomotika, S., and Tamada, K. Studies on two-dimensional transonic flows of compressible fluid. III. Quart. Appl. Math. **9**, 129–147 (1951).

This is part III of a series of studies in two-dimensional transonic flows [same Quart. **7**, 381–397 (1950); **8**, 127–136 (1950); these Rev. **11**, 275; **12**, 138]. Here a different approximation to the fundamental equations for flow in the hodograph plane is proposed. It is found that if the local supersonic Mach number does not exceed 1.2, the approximation may be expected to yield useful results. By means of this new simplification, the particular integrals of the differential equation reduce to products of Bessel and trigonometric functions. By a well-known property of Bessel functions, the authors are able to construct a solution in closed form, which, when transformed back to physical plane, corresponds to a flow past a 10 per cent symmetric airfoil with a cusped trailing edge. The flow patterns have been calculated for three free-stream Mach numbers: 0.717, 0.745 and 0.752. For 0.717, critical speed is just reached at the minimum pressure point. The second case has a finite local supersonic region after which there follows an extremely large pressure gradient. Due to this high pressure gradient, a limiting line appears at the slightly higher Mach number 0.752. Between the two extreme cases, the boundary shape, however, has changed only very slightly.

Y. H. Kuo (Ithaca, N. Y.).

Germain, Paul. Application de l'approximation homographique à l'étude des écoulements transsoniques. C. R. Acad. Sci. Paris **232**, 1811–1813 (1951).

By approximating the coefficient $k(\sigma)$ in the hodograph equation $k(\sigma)\psi_{\theta\theta}+\psi_{\sigma\sigma}=0$ by $\sigma(1+\sigma)^{-1}$, the author obtains a fundamental solution of the approximate equation. By superposition, general solutions in integral forms can be constructed. In a more general manner, the same procedure has recently been exploited by Tomotika and Tamada [see the preceding review]. *Y. H. Kuo* (Ithaca, N. Y.).

Wood, George P., and Gooderum, Paul B. Method of determining initial tangents of contours of flow variables behind a curved, axially symmetric shock wave. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2411, 44 pp. (1951).

If the pressure, density, and Mach number are known on a non-characteristic curve in a plane through the axis of symmetry of an axisymmetric flow, then their gradient vectors can be determined. By geometrical methods the authors have derived formulae for the slopes of the contours of these functions immediately behind a given shock, ahead of which the flow is known. An application to the density field about a sphere at $M=1.62$, experimentally determined from an interferogram, shows that the theoretical slopes at

the shock fair well into the measured constant density contours. *J. H. Giese* (Havre de Grace, Md.).

Cabannes, Henri. Le problème de l'onde de choc détachée pour les écoulements de révolution. Recherche Aéronautique no. 21, 3–7 (1951).

The axisymmetrical flow resulting from the passage of a uniform supersonic stream through a stationary detached shock, whose equation in cylindrical polar coordinates is

$$x=\frac{r^2}{2R}+\lambda_2\frac{r^4}{4R^3}+\lambda_3\frac{r^6}{6R^5}+\cdots,$$

so that R is its radius of curvature on the axis, is investigated by the method of Taylor expansions. The equations of inviscid motion with entropy conserved along streamlines, and the Rankine-Hugoniot shock conditions, are satisfied up to and including the fourth powers of distances r or x from the vertex of the shock. A dividing streamline representing an obstacle that could have caused the detached shock, is determined to this approximation. The said obstacle is hollow (like a ducted body) if $\lambda_2>\lambda^*(M)$, a certain increasing function of M which for example is 4.65 for $M=2$. (Clearly large λ_2 means a rather flat-fronted shock.) The surface is not hollow, and meets the axis of revolution if $\lambda_2<\lambda^*(M)$. In this case the distance h of the stagnation point from the shock vertex, and the radius of curvature $\mathfrak{R}$ of the body at the stagnation point, are predicted. The former is in good agreement with experiment, but the latter is only obtained to a first approximation and for it the agreement is, understandably, poorer.

When the method is inverted in order to infer shock shape from body shape, the author finds that more than one solution to the problem appears to exist, which offers an interesting analogy to the state of affairs for an attached shock. Thus, if the body is given by the equation

$$x=h+\frac{r^2}{2\mathfrak{R}}+\tau_2\frac{r^4}{4\mathfrak{R}^3}+\cdots,$$

then τ_2 is a one-valued function of h/R, but h/R is a three-valued function of τ_2 for a certain range of positive values of τ_2. The author states that the largest of the three branches of h/R is that observed experimentally, and explains this by saying that it is the regime with least entropy increment. But he does not discuss what happens when τ_2 is below the range of positive values mentioned above; the only branch of h/R which then exists is then discontinuous with that mentioned before. This leads to the possibility of shock shapes which are discontinuous functions of body shape or Mach number, with the likelihood of oscillation between them near the transition point, which should be further investigated, theoretically and experimentally.

M. J. Lighthill (Manchester).

Cabannes, Henri. Détermination de l'onde de choc devant un obstacle de révolution lorsque la vitesse à la pointe sur l'obstacle est subsonique. C. R. Acad. Sci. Paris 233, 354–356 (1951).

Whitham, G. B. The propagation of spherical blast. Proc. Roy. Soc. London. Ser. A. 203, 571–581 (1950).

The problem of the paper and the method used for its solution are similar to those of the author's study of supersonic flow past a body of revolution [same Proc. Ser. A. 201, 89–109 (1950); these Rev. 12, 298]. The mathematical principle was described by Lighthill [Philos. Mag. (7) 40, 1179–1201 (1949); these Rev. 11, 518]. If r is the distance from the center of the spherical blast and z is the characteristic parameter constant on outgoing characteristics, the method is to expand velocity and time t in a descending series of r, appropriate for large r, the coefficients of which are functions of z. By fitting the approximate shock conditions, the equation of the leading shock is found to be

$$\text{(A)}\qquad a_0t=r-b\log^{\frac{1}{2}}r-b_1-b_2\log^{-\frac{1}{2}}r+O(\log^{-1}r)$$

where a_0 is the velocity of sound in the undisturbed air and b, b_1, and b_2 are arbitrary constants. This general theory is used to modify the solution of the simple linear theory so that it is uniformly valid at all distances from the origin. Then the constant b in (A) can be easily computed from the given characteristics of the blast. It is shown that generally there is a second shock behind the first shock and the pressure at a point between shocks falls approximately linearly with time at a rate

$$\{2\gamma/(\gamma+1)\}a_0r^{-1}\log^{-\frac{1}{2}}r\text{ atm./sec.}$$

H. S. Tsien (Pasadena, Calif.).

Chien, Wei-Zang. Symmetrical conical flow at supersonic speed by perturbation method. Eng. Rep. Nat. Tsing Hua Univ. 3, no. 1, 1–14 (1947). (English. Chinese summary)

This is a report on a calculation, by a method of successive approximations, of symmetrical flow over a right circular cone. First the complete, nonlinear differential equation for the velocity potential is written, together with the shock-wave conditions and the condition of tangential flow at the cone. Use is made of the conical symmetry. The potential is then sought in the following form:

$$\varphi=V\{x+\epsilon^2\varphi_1+\epsilon^4\ln(2/\epsilon)\varphi_2+\epsilon^4\varphi_3+\cdots\},$$

where ϵ is proportional to the tangent of the cone angle and the φ_i are independent of ϵ. They are calculated successively, φ_1 being the familiar linearized-theory result. The consistent approximations at each step are made in the differential equation and the boundary conditions. The calculations are carried through φ_3 in this paper, and results are compared with exact (Taylor-Maccoll) results for cones up to 30° semivertex angle.

This paper was presented in the U. S. A. in 1946, and therefore actually antedates similar investigations by Lighthill [Quart. J. Mech. Appl. Math. 1, 309–318 (1948); these Rev. 10, 413] and Broderick [ibid. 2, 98–120, 121–128 (1949); these Rev. 10, 643, 644]. The reviewer finds that there are appreciable discrepancies in the comparable results of the present paper and those of Lighthill and Broderick, which agree with one another, but has not taken the time to determine the source of this disagreement.

W. R. Sears (Ithaca, N. Y.).

Kawaguchi

Kawaguchi, Mitutosi. Application of the M^2-expansion method to the compressible flow past an elliptic cylinder. J. Phys. Soc. Japan 6, 168–174 (1951).

The terms of order M^4 in the Janzen-Rayleigh expansion for symmetrical flow past an elliptic cylinder are found. The velocity distribution on the cylinder, on the Janzen-Rayleigh method including these terms, is compared numerically with that obtained on various other methods.

M. J. Lighthill (Manchester).

Harmon, Sidney M. Method for calculating downwash field due to lifting surfaces at subsonic and supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2344**, 30 pp. (1951).

The author's summary is as follows: A method utilizing source singularities is presented for obtaining the linearized downwash field due to lifting wings at subsonic and supersonic speeds. The method is applied to derive generalized formulas for the downwash field due to uniformly loaded swept and rectangular wings at subsonic and supersonic speeds. The utilization of these formulas to obtain the downwash due to wings of arbitrary loading is indicated.

E. Reissner (Cambridge, Mass.).

Kuo, Yung-Huai. Two-dimensional transonic flow past airfoils. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2356**, 48 pp. (1951).

This is a continuation of earlier work by Tsien and Kuo [same Tech. Notes, no. **995** (1946); no. **1445** (1948); these Rev. **8**, 237; **11**, 223]. Let W be the complex velocity potential of an incompressible flow about some profile B, with velocity one at infinity. Let $dW/dz=w=qe^{-i\theta}$, and suppose $W(w)=-\sum_1^\infty A_n w^n$, $|w|<1$, near some stagnation point of B. A stream function for a compressible flow over a distorted body B' can be constructed by multiplying each term in the imaginary part of $W(w)$ by a suitable hypergeometric function $F_n(\tau)$, where $\tau=\frac{1}{2}(\gamma-1)q^2c_0^{-2}$ and c_0 is the stagnation speed of sound. To facilitate analytic continuation beyond $|w|=1$, the author adds a series of terms $\tau^{\frac{1}{2}m}F_m e^{im\theta}$ etc. to the compressible stream function. Instead of applying the method developed by Tsien and Kuo, the author performs the analytic continuation by a method due to Lighthill [Proc. Roy. Soc. London. Ser. A. **191**, 352–369 (1947); these Rev. **9**, 391] and Cherry [same Proc. **192**, 45–79 (1947); these Rev. **9**, 544]. This involves expanding some of the F_n into series and changing the order of summation in the double series in the compressible stream function to obtain transformed series which converge for $|w|>1$. The author has applied the outlined procedure to determine approximate stream and potential functions for the symmetrical compressible flow about a distorted approximation to a thin Joukowski airfoil defined as follows. For a Joukowski airfoil $W(\zeta)=\zeta+\epsilon+(1+\epsilon)^2/(\zeta+\epsilon)$ and

$$w(\zeta)=\zeta^2(\zeta+1+2\epsilon)/(\zeta+1)(\zeta+\epsilon)^2,$$

where $z=\zeta+1/\zeta$. If second order terms in ϵ are neglected hereafter, then

$$W=\zeta+\epsilon+\tfrac{1}{2}\epsilon^{-1}(1+2\epsilon)\log(1+2\epsilon\zeta^{-1})$$
$$=\zeta+\epsilon+(1+2\epsilon)\zeta^{-1}-\epsilon\zeta^{-2}+O(\epsilon^2),$$

and $w=\zeta(\zeta+1+2\epsilon)/(\zeta+1)(\zeta+2\epsilon)$. A similar approximation to a thin Joukowski airfoil at an angle of attack of order ϵ has also been considered.

J. H. Giese.

Lomax, Harvard, Heaslet, Max. A., and Fuller, Franklyn B. Three-dimensional unsteady lift problems in high-speed flight—the triangular wing. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2387**, 62 pp. (1951).

The authors consider first the transient loading of the triangular wing with leading edges outside the Mach cone, a problem solved earlier by Strang [Proc. Roy. Soc. London. Ser. A. **202**, 54–80 (1950); these Rev. **12**, 216] and the reviewer [U. S. Naval Ordnance Test Station, Inyokern, Calif. Nav. Rep. **1235** (1950)]. In addition, the loading over a portion of the wing with leading edges inside the Mach cone is obtained, but the solution for that portion involving mixed boundary conditions is not found. The very narrow wing is treated approximately by neglecting the streamwise derivatives in the potential equation. [In the opinion of the reviewer, the approximation is inconsistent, and the time derivatives also should be neglected, in which case nonstationary effects appear only in the boundary conditions and Bernoulli's equation.]

J. W. Miles.

Chang, Chieh-Chien. Transient aerodynamic behavior of an airfoil due to different arbitrary modes of non-stationary motions in a supersonic flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2333**, 65 pp. (1951).

The author extends his earlier work on the transient motion of a two-dimensional airfoil [J. Aeronaut. Sci. **15**, 635–655 (1948)] to more general motions, using Duhamel superposition. Particular attention is paid to abruptly started sinusoidal motion. The reviewer notes that the function $C(\beta, M)$ introduced by the author can be reduced to the well known flutter function $f_0(\lambda, M)$ [L. Schwarz, Luftfahrtforschung **20**, 341–372 (1944); these Rev. **5**, 238] according to

$$C(\beta, M)=1-i\,(M^2-1)^{-\frac{1}{2}}\beta M f_0[\beta M^2(M^2-1)^{-1}, M].$$

J. W. Miles (Los Angeles, Calif.).

Ferrari, Carlo. Sulla determinazione di alcuni tipi di campi di corrente ipersonora. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 277–283 (1950).

The author considers the problems of the flow field of an infinite set of thin wings, arranged periodically tip to tip, and the thin body of revolution (very briefly). The wave equation for the perturbation potential φ, $(1-M_\infty^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}$, where (x, y, z) are the coordinates in the direction of flow, spanwise and surface normal respectively, is solved by a Fourier analysis with respect to the spanwise direction. For example, when the wing semi-spacing is 1.0 and the wing is non-lifting, $\varphi=\sum\varphi_m(x, z)\cos\frac{1}{2}\pi my$, where φ_m satisfies the telegraphist's equation. Consequently

$$\varphi_m(x, z)=\int_0^{x-z(M_\infty^2-1)^{\frac{1}{2}}} h_m(x')$$
$$\times J_0(\tfrac{1}{2}m\pi[(x-x')^2-(M_\infty^2-1)z^2]^{\frac{1}{2}}/[M_\infty^2-1]^{\frac{1}{2}})dx'$$

and J_0 is the Bessel function of zeroth order. The set of constants $\{h_m\}$ are simply related to the thickness distribution. The body of revolution is likewise analyzed by Fourier expansion with respect to the meridional angle. The author refers to his discussion of wing-body interference [J. Aeronaut. Sci. **16**, 542–546 (1949)] as an application of the method.

F. E. Marble (Pasadena, Calif.).

Teofilato, Pietro. Applicazione del metodo delle caratteristiche alla corrente supersonica vorticosa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 124–129 (1951).

The author describes a step-by-step procedure based on a Mach characteristic net for calculating the entropy and the gradient of the (mass flow) stream function for steady plane flow. He shows that the local error in the entropy so determined is of the second order in the grid size, whereas in Ferri's method of characteristics [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1135** = Tech. Rep., no. **841** (1946); these Rev. **8**, 106; **10**, 491] the error is of first order.

J. H. Giese (Havre de Grace, Md.).

Stocker, P. M. Supersonic flow past bodies of revolution with thin wings of small aspect ratio. Aeronaut. Quart. 3, 61–79 (1951).

Ward [Quart. J. Mech. Appl. Math. 2, 75–97 (1949); these Rev. 10, 644] worked out the supersonic small-perturbation flow past slender pointed bodies in a way that makes it applicable to a wide class of bodies of various cross-sections. His method resembles "slender-wing theory," since the potential of the flow is identified with that of a two-dimensional incompressible flow in transverse planes, but a term that would be a constant in such a two-dimensional flow must, in general, be evaluated with the aid of the more complete theory. Ward gave a formula for this term. Here his method is applied to three cases of slender bodies of revolution with thin wings. In one case, where the incidence is zero, the pressure distribution on the wing is calculated. In the other cases lift and drag are obtained.

W. R. Sears (Ithaca, N. Y.).

Burns, J. C. Airscrews at supersonic forward speeds. Aeronaut. Quart. 3, 23–50 (1951).

Let $\mathbf{r}=(x, y, z)$ be the rectangular coordinate vector relative to axes fixed in an airscrew which rotates with constant angular velocity $\mathbf{\Omega}=(-\Omega, 0, 0)$ while the origin is translated with constant velocity $\mathbf{U}=(U, 0, 0)$. Assume the airscrew is bounded by $z=y\tan[(\Omega+\omega)x/U]\pm\epsilon f(x, y)$, whose mean surface would produce no disturbance at angular velocity $\Omega+\omega$, where $\omega>(<)0$ for windmills (propellers). Let $\mathbf{U}+\operatorname{grad}\varphi(x, y, z)$ be the instantaneous velocity in linearized irrotational flow about this airscrew, relative to stationary axes instantaneously coincident with the $\mathbf{r}$-axes. The disturbance potential satisfies

$$\begin{aligned}(1-M^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz} &=\Omega^2a_0^{-2}(z^2\varphi_{yy}-2yz\varphi_{yz}+y^2\varphi_{zz}-y\varphi_y-z\varphi_z)\\ &\quad+2\Omega Ua_0^{-2}(y\varphi_{xz}-z\varphi_{xy})\end{aligned}$$

with $M=U/a_0$, subject to the boundary condition

$$\partial\varphi/\partial n=-(\mathbf{U}-\mathbf{\Omega}\times\mathbf{r})\cdot\mathbf{n}^*,$$

where $\mathbf{n}^*$ is a unit normal to the airscrew, and the normal derivative is actually evaluated on $z=y\tan\Omega x/U$. Now set $\varphi=\varphi_1+\varphi_2$, where φ_1 corresponds to the airscrew $z=y\tan[(\Omega+\omega)x/U]$, and φ_2 to $z=y\tan(\Omega x/U)\pm\epsilon f(x, y)$. Expand φ_1 and φ_2 in series of the form

$$\varphi_1=\varphi_{0(1)}+(\Omega b/a_0)\varphi_{1(1)}+(\Omega b/a_0)^2\varphi_{2(1)}+\cdots,$$

and simplify boundary conditions by neglecting ϵ^2, ω^2, $\Omega^2\epsilon$, $\Omega^2\omega$, and Ω^3, but not Ω^2. The determination of $\varphi_{0(1)}$ etc. is reduced eventually to constructing odd solutions of $(1-M^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0$ by the method of G. N. Ward [Quart. J. Mech. Appl. Math. 2, 136–152 (1949); these Rev. 11, 64] and even solutions by the method of A. E. Puckett [J. Aeronaut. Sci. 13, 475–484 (1946); these Rev. 8, 109].

For $f=4(cx-x^2)/c$ and plan form $|y|\leqq b$, $0\leqq x\leqq c$ in the (x, y)-plane, the parts of $\varphi_{0(1)}$, $\varphi_{1(1)}$, $\varphi_{0(2)}$, $\varphi_{1(2)}$ independent of the blade tips have been calculated, as well as the corresponding axial torque and drag. At $M=\sqrt{2}$ the variations of the powers produced by the windmill and absorbed by drag have been studied as functions of blade thickness, aspect ratio, and $\Omega b/a_0$, and structural limitations on blade thickness have been roughly estimated.

J. H. Giese.

Vialar, J. Analyse du champ de divergence. J. Sci. Météorologie 3, 37–56 (1951). (French. English, French, and Spanish summaries)

An expression for the horizontal velocity divergence is derived from the gradient wind equation. Its terms express the effects of varying latitude, curvature and density as well as that of confluence. Reasonable maximum values of the terms are estimated, with the result that the curvature term can be the most important. The values of the total divergence may be as large as 5×10^{-5} sec^{-1}. Finally, the modification of the theory due to use of constant pressure rather than constant level charts is discussed.

H. Panofsky.

Carter, A. H., and Williams, A. O., Jr. A new expansion for the velocity potential of a piston source. J. Acoust. Soc. Amer. 23, 179–184 (1951).

According to A. Schoch [Akustische Z. 6, 318–326 (1941); these Rev. 8, 239] the Rayleigh surface integral, giving the velocity potential ϕ for a plane circular piston source of strength w in an infinite rigid baffle, is reducible to a line integral, viz.

$$(*)\qquad \phi=\frac{w}{ik}e^{-ikz}-\frac{w}{2\pi ik}\int_0^{2\pi}e^{-ikr_1}d\theta,\qquad x<a,$$

where k is the wave number, x and z cylindrical coordinates with origin at the centre of the piston (of radius a) and

$$r_1^2=z^2+a^2[b\cos\theta+(1-b^2\sin^2\theta)^{\frac12}]^2,\quad b=x/a.$$

A paraxial approximation to ϕ is given which extends earlier results of various authors. A rigorous calculation of ϕ is obtained by employing Lommel's Hankel function expansion [cf. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1944, sec. 5.22; these Rev. 6, 64] of the integrand of (*). The authors' result is

$$(**)\qquad \phi=-\frac{w}{ik}\sum_{m=1}^{\infty}\frac{(-1)^m(ka)^m}{2^m m!}\left(\frac{a}{z}\right)^m\left(\frac{\pi}{2}kz\right)^{\frac12}H^{(2)}_{m+\frac12}(kz)f_{2m},$$

where

$$f_{2m}=\frac{1}{\pi}\int_0^{\pi}[b\cos\theta+(1-b^2\sin^2\theta)^{\frac12}]^{2m}d\theta=F(1-m, -m; 1; b^2).$$

Numerical values of f_{2m} are tabulated for $b=0(0.1)1$, $m=1(1)8$ [5D]. The series (**) is evaluated for two cases: $ka=10$, $z=10\,a$ and $ka=50$, $z=50\,a$. [The authors wish to correct their statement as to the convergence of (**): the domain of convergence is given by $z>x+a$ instead of $z>a$. Misprints in the footnotes to p. 182: first line, 1015 should read 1012; second line, 1946 instead of 1934. Third line should read "† In reference 12 the eighth degree polynomial f_{16} appears to be computed incorrectly".]

C. J. Bouwkamp (Eindhoven).

Blohincev, D. I. The theory of moving sources and sound receivers. Učenye Zapiski Moskov. Gos. Univ. Fizika. 134, kniga 5, 134–144 (1949). (Russian)

Elasticity, Plasticity

★Lehnickii, S. G. Teoriya uprugosti anizotropnogo tela. [Theory of Elasticity of an Anisotropic Body]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 299 pp.

This monograph is principally concerned with a systematic presentation of recent contributions in the domain of linear anisotropic theory of elasticity made in the USSR.

Chapter 1 (66 pp.) contains a discussion of the basic concepts of anisotropic theory of elasticity and culminates in the formulation of general equations governing the behavior of anisotropic media. Chapter 2 (20 pp.) reviews the simplest cases of stress distribution in anisotropic rods and plates, such as considered by W. Voigt, Lehrbuch der Kristallphysik [Teubner, Leipzig and Berlin, 1910]. The remaining four chapters (206 pp.) contain a condensed summary of recent contributions. The state of stress in a homogeneous anisotropic cylinder subjected to the action of forces in the plane normal to the axis of the cylinder, and not varying along the axis, is analyzed in chapters 3 and 4. The special case of cylindrically anisotropic media is presented in some detail. In chapter 5, the equilibrium of a cantilever beam of uniform cross section under a load on its free end is discussed; several problems involving symmetric deformation and torsion of bodies of revolution are treated in chapter 6.

The monograph contains no discussion of deflection and stability of anisotropic plates, but these problems have been treated in the author's earlier book, Anisotropic Plates [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1947; these Rev. 10, 415]. A stress analyst working with anisotropic media will find this book highly useful because of many explicit design formulas included in it. The bibliography contains 56 items, 45 of which are in the Russian language.

I. S. Sokolnikoff (Los Angeles, Calif.).

Aržanyh, I. S. Integral equations for the representation of the vector of translation, spatial dilatation, and rotation of an elastic body. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 387–391 (1951). (Russian)

The displacement vector $u(x_1, x_2, x_3, t)$ of a three-dimensional homogeneous isotropic elastic body occupying a domain Q (which is either the interior or the exterior of a closed surface S) satisfies in Q the following system of equations

$$\mu\nabla^2 u+(\lambda+\mu)\text{ grad div } u+\rho b=\rho\partial^2 u/\partial t^2,$$

where λ and μ are Lamé's constants, ρ is the density, and b is the body force per unit mass. In the first boundary value problem, the displacement u is assigned on S, while in the second boundary value problem the surface traction (n is the outer normal to S and $\times$ denotes the vector product)

$$2\mu\partial u/\partial n+\lambda n\text{ div } u+\mu n\times\text{rot } u=F_n$$

is assigned on S. In the exterior problem (Q is the exterior of S) u is required to be regular at infinity. In the present paper the author obtains integral equations for the Laplace transforms of the displacement u, the dilatation div u, and rot u, in terms of the usual Green's function for the Dirichlet problem in the case of the first boundary value problem, and Neumann's function in the case of the second boundary value problem.

J. B. Diaz (College Park, Md.).

Kondo, Kazuo. Mathematical analyses of the yield point. III. Isotropic stress. J. Jap. Soc. Appl. Mech. **4**, 35–38 (1951).

Continuing his researches on the theory of yield [same J. 3, 184–188 (1950); these Rev. 12, 771] the author claims to show that according to his theory yield is impossible under a hydrostatic stress. *C. A. Truesdell.*

Graffi, Dario. Su alcune questioni di elasticità ereditaria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 25–30 (1951).

In Volterra's accumulative theory of elasticity it is possible to express the strain $e^i{}_j$ in the form

$$e^i{}_j(t)=\frac{1+\sigma}{E}t^i{}_j(t)-\frac{\sigma}{E}t^k{}_k(t)\delta^i{}_j+\int_0^t g(t,\tau)t^i{}_j(\tau)d\tau-\delta^i{}_j\int_0^t h(t,\tau)t^k{}_k(\tau)d\tau,$$

where σ and E are constants, h and g are given functions. The author seeks conditions under which for given circumstances the stress in a simply-connected body shall be the same as that given by the classical infinitesimal theory. The resulting displacements in the two theories are of course quite different nevertheless. He finds that, given fixed extraneous forces and surface loads, it is sufficient that (1) $t^k{}_k$ in the classical solution be linear in x, y, z, or that (2) $\sigma g=(1+\sigma)h$. The first result includes many deformations of practical interest. The second result is extended to the case when the load or a part of the boundary surface is prescribed, while on the remainder the displacement is zero. The author shows also that in any case of plane stress subject to no extraneous force the stresses in Volterra's theory are the same as those of the infinitesimal theory.

C. A. Truesdell (Bloomington, Ind.).

Bernabini, Maria. Sull'integrazione delle equazioni dell' elasticità piana in coordinate curvilinee. Rend. Accad. Sci. Fis. Mat. Napoli (4) **17**, 6 pp. (1950) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **292** (1951).

The author considers the problem of plane elasticity in an isothermal coordinate system (ξ, η) so that $x+iy=g(\xi+i\eta)$, where $g(\zeta)$ is analytic. She shows that the general solution of the problem can then be expressed simply in terms of $|g'(\zeta)|$ and two biharmonic functions whose Laplacians are conjugate harmonic functions. The result is valid in multiply-connected as well as simply-connected regions. As an application she solves a problem concerning an eccentric circular annulus.

C. Truesdell (Bloomington, Ind.).

Guerra, Guido. Un metodo grafico numerico per la risoluzione dei problemi di elasticità piani studiati con rilevamenti fotoelastici. Ricerca, Napoli 1, nos. 2–3, 57–74 (1950).

The author discusses a new method for computing the stress fields in simply connected thin plates under the influence of distributed or concentrated loads parallel to their faces. The basic data, obtainable by the usual photoelastic procedures, are taken to be the isoclinics and the maximum shear at all points, including the boundaries. The author's method is a combination of two well-known procedures, i.e., a method for determining p, the first stress invariant, by solving the corresponding Dirichlet problem numerically, and one based on the numerical integration of the Maxwell-Lamé form of the equations of equilibrium. No actual illustrations of the advantages of the method proposed over conventional ones are presented in this paper.

A. W. Sáenz (Bloomington, Ind.).

Grinberg, G. A. On the solution of the plane problem of the theory of elasticity and of the problem of bending of a thin plate with clamped contour. Doklady Akad. Nauk SSSR (N.S.) 76, 661–664 (1951). (Russian)

It is well known that plane problems of elasticity, when the stresses are given on the boundary, as well as the problem of the deflection of thin homogeneous isotropic elastic plates lead to the following boundary value problem: to determine a function ψ such that

$$\Delta^2\psi=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi=\Phi,$$

in a plane open set, where Φ is a given function, and further the values of ψ and $\partial\psi/\partial n$ are prescribed on the boundary. The author outlines a formal method for the approximate solution of this boundary value problem, based on the fact that if θ is any function (to be chosen in applications) such that $\Delta\theta=\Phi$, then $\Delta[\Delta\psi-\theta]=0$, and that $\Delta\psi-\theta$ may be approximated by linear combinations of chosen harmonic functions. The procedure is applied to the deflection of a uniformly loaded clamped square plate. *J. B. Diaz.*

Šerman, D. I. On the stresses in a heavy half-plane weakened by two circular openings. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 297–316 (1951). (Russian)

An approximate solution of the elastostatic plane problem for a homogeneous, isotropic half-plane weakened by two circular holes, whose centers lie on a line parallel to the boundary of the half-plane, is obtained under the assumption that the openings are not too near the boundary. The boundaries of the holes are assumed to be free of external loads and the elastic half-plane, in addition to its own weight, is subjected to the action of a uniform tensile stress applied at infinity. The solution is obtained by determining the Mushelišvili potentials from certain integrals of Cauchy's type. The problem is illustrated by a numerical example for the case in which the radius of one hole is three times the radius of the other, and the distance between the holes is equal to the smaller radius. *I. S. Sokolnikoff.*

Ling, Chih-Bing. On torsion of prisms with longitudinal holes. Quart. Appl. Math. 9, 247–262 (1951).

This paper presents a method of solution, called the method of images, for the torsion of prisms having one or more longitudinal holes. The method is applicable to prisms of the following four, and only four sections: a rectangle, an equilateral triangle, an isosceles triangle and a 30°-60°-90° triangle. These four sections form a group by themselves. The solution is obtained by adding to the known solution of a corresponding solid prism without holes a system of harmonic functions which vanish on the entire external boundary of the given section, and besides possess a singularity at the centre of each hole. Such a series of functions may be constructed from Weierstrass' Sigma function and its allied functions.

The solution is illustrated by applying it in detail to a rectangular prism having a central longitudinal circular hole. The system of functions added are expressed in terms of polar coordinates referred to the centre of the hole as origin, and having arbitrary coefficients. These coefficients are determined from the boundary condition on the edge of the hole by means of an iterative process. The validity of the solution depends, of course, on the convergence of the series involved, but physical considerations indicate convergence under certain conditions. The torsional modulus and stress components are expressed in terms of the same functions and numerical results are given for the special case of the square prism. *R. M. Morris* (Cardiff).

Radok, J. R. M. Vibrations of a swept box. Coll. Aeronaut. Cranfield. Rep. no. 47, 12 pp. (1 plate) (1951).

A model of a swept-back wing is considered having a uniform rectangular box section at all sections along the span. The plan is a parallelogram with wing root and wing tip planes vertical and parallel to the flight direction. Deformation is described by rotations of the wing sections about the wing axis and the direction of flight. In terms of these and their derivatives, and an auxiliary dependent variable, the vertical displacement of the centre of a section, the kinetic and potential energies are expressed for substitution in the variational equation representing Hamilton's Principle. The integro-differential equations obtained are transformed into a system of homogeneous Fredholm integral equations for a normal mode vibration, and an approximate numerical method of solution is indicated. *E. H. Lee.*

Satô, Yasuo. Boundary conditions in the problem of generation of elastic waves. Bull. Earthquake Res. Inst. Tokyo 27, 1–9 (1949).

This paper concerns the dynamic elastic problem for a solid sphere with time dependence occurring solely as a sinusoidal factor. The equations of motion are written down in spherical coordinates, and it is shown that solutions involve displacements which are the sum of three terms: the gradient of a function, the curl of a vector function with one non-zero component, and the double curl of such a vector function. By expressing these functions as series involving spherical harmonics, the determination of coefficients is detailed for an arbitrary prescribed distribution of the three components of displacement on the spherical surface. A similar solution is shown to apply for prescribed surface tractions, and for certain initial value problems. It is stated that previously only restricted displacement conditions have been treated, for example radial displacement prescribed to be non-zero with the other components zero.

E. H. Lee (Providence, R. I.).

Satô, Yasuo. Mathematical study of the propagation of waves upon stratified medium. I. Bull. Earthquake Res. Inst. Tokyo 26, 1–4 (1948).

Approximate expressions of definite integrals in (a, b) with the integrands $f(t)\cos xt$ and $f(t)\sin xt$ for $|x|\to\infty$, are deduced with the aid of integrations by parts. These expressions are valid under the following very restrictive conditions: $f(t)$ has a derivative of some order m, $f^{(m)}(t)$, which is continuous in $a+\epsilon\leqq t\leqq b$ $(\epsilon>0)$ and the analytic function $\varphi(z)=zf(z^2+a)=(t-a)^{\frac{1}{2}}f(t)$, where $z=(t-a)^{\frac{1}{2}}$, is regular around the origin $z=0$ $(t=a)$. The author claims that these elementary results are important for the study of propagation of waves upon the surface of elastic bodies.

E. Kogbetliantz (New York, N. Y.).

Satô, Yasuo. Rayleigh waves propagated along the plane surface of horizontally isotropic and vertically aeolotropic elastic body. Bull. Earthquake Res. Inst. Tokyo 28, 23–29 (1950). (English. Japanese summary)

The main result of this paper is that one and only one sort of Rayleigh wave exists in a horizontally isotropic but vertically anisotropic semi-infinite elastic medium

E. Kogbetliantz (New York, N. Y.).

Takahashi, Takehito, and Satô, Yasuo. On the theory of elastic waves in granular substance. I. Bull. Earthquake Res. Inst. Tokyo **27**, 11–16 (1949).

A granular substance is considered as a space lattice of smooth elastic spheres. Initial pressure produces compression at the points of contact, and the equations of motion for small changes of the contact force are developed. A wave solution is considered, and for wave-length large compared with the diameter of the spheres, the system can be considered as an equivalent continuous elastic body for which the elastic constants are determined. Appropriate average values of these are taken to allow for the random orientation of the lattice in the granular substance. Numerical results for wave velocities in sand are computed and found to agree with measured values. *E. H. Lee* (Providence, R. I.).

Takahashi, Takehito, and Satô, Yasuo. On the theory of elastic waves in granular substance. II. Bull. Earthquake Res. Inst. Tokyo **28**, 37–43 (1950). (English. Japanese summary)

This paper contains more detailed consideration of points arising in the paper reviewed above. The wave velocities for the lattice of smooth spheres are obtained for simple cubic, body-centered cubic, and face-centered cubic lattices. Details are given of the determination of the mean elastic constants for a granular substance with random lattice orientation assuming uniform strain. This is carried out by analysing the matrix representing rotation into basic components to simplify the integration over all possible lattice rotations. *E. H. Lee* (Providence, R. I.).

Mandel, J. Essai sur la mécanique physique des pseudosolides. Ann. Ponts Chaussées **120**, 245–312 (1950).

The term "pseudosolide" in the title of this paper refers to an aggregate of solid grains the voids between which are filled with a fluid (water) and a gas (air). The purpose of the expository paper is to present the mechanical and thermodynamical concepts which have been introduced in the study of pseudo-solids, particularly in soil mechanics. The first part of the paper is concerned with the equilibrium and motion of the fluid in a pseudosolid. The second part deals with the stresses in a pseudosolid, and the mechanical action of the fluid on the structure formed by the solid grains. *W. Prager* (Providence, R. I.).

✶Sokolovskiĭ, V. V. Teoriya plastičnosti. [Theory of Plasticity]. 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 396 pp.

This, the second, edition of the book presents a thorough revision of both the contents and the development of the subject found in the first edition [cf. these Rev. **8**, 545]. The development of the subject is more deliberate and better organized. Familiar ideas of the theory of elasticity are more frequently used as an introduction to plastic behavior. On the whole, the book is now more readable.

The following new topics are discussed: simple loading of bodies and connection between the theory of elastic-plastic deformations and the theory of plastic flow; unloading in the theory of elastic-plastic deformations; trigonometric representation of the compounds of stress and strain, torsion of conical bars with nonlinear hardening; new methods of solutions of the plane strain problem; a new analogy for the plane elastic-plastic problem; plane strain state for a generalized stress-strain law; plane equilibrium of a plastic wedge under variable load; plane and axially symmetric equilibrium of a plastic mass between rigid walls; uniform rotation of a disk and bending of a round plate with nonlinear hardening.

However, the improvements over the first edition are offset by a number of serious disadvantages. The important developments of the mathematical theory of plasticity outside the USSR during the last three years are largely neglected. For example, the work of Hill and Lee in England, and of Prager, Drucker, and Greenberg in this country is not mentioned. Among the most important omissions the following should be mentioned: (1) Discussion of the inherent difference between finite and incremental stress-strain laws and realization of the fact that finite laws violate certain physical requirements. (2) Realization of the fact that most practical boundary value problems are of mixed character involving surface stresses and surface displacements. Accordingly, the emphasis on statically determinate problems does not seem justified as was shown by the work of Hill and Lee. (3) Discussion of discontinuous stress and velocity fields which often represent the only admissible solution and have also proved useful for approximation purposes. (More recently, such fields have been used successfully in the construction of bounds for the safety factor; since much of this work is not yet published in form other than mimeographed reports, however, the author can not be blamed for ignoring it.)

The chapter headings are as follows: (I) Theories of plasticity; (II) Fundamental equations of plastic equilibrium; (III) Simple problems of elastic-plastic equilibrium; (IV) Plastic torsion of rods; (V) Plane plastic state; (VI) Stress distribution in plastic zones surrounding holes; (VII) Indentation of a plastic body; (VIII) Compression and extrusion of a plastic sheet; (IX) Plane stress state; (X) Plane strain state for a general law of plasticity; (XI) Plane equilibrium of a plastic wedge; (XII) Plane and axially symmetric equilibrium of a plastic mass between rigid walls; (XIII) Elastic-plastic bending of beams and plates; (XIV) Elastic-plastic bending of round plates and rings. *H. I. Ansoff* (Santa Monica, Calif.).

Hill, R. On the state of stress in a plastic-rigid body at the yield point. Philos. Mag. (7) **42**, 868–875 (1951).

The paper is concerned with the mechanical behavior of a rigid-plastic body under a given loading schedule. For sufficiently small loads, the stress distribution is supposed to be that in an elastic body, but the shear modulus is assumed to be indefinitely large so that the body remains rigid. Even after the body first becomes locally plastic at the "elastic limit", the rigidity of the adjacent "elastic" material frequently prevents the occurrence of plastic deformation in the plastic region and the body remains entirely rigid although it is partly plastic. It is only when the plastic region attains a certain critical size or configuration determined by the geometry of the problem that plastic deformation becomes possible. The author defines as the "yield point" of the body that stage of the loading schedule at which the first plastic deformation occurs.

The mathematical discussion is based on the assumption of a convex yield function which also serves as plastic potential. As a typical boundary value problem the author considers the case where the surface tractions are prescribed on a part S_T of the surface and the velocities on the remainder S_V. For the purpose of establishing a restricted uniqueness theorem it is assumed that for given boundary conditions of this type two stress fields and the associated velocity fields can be found which satisfy all conditions of the prob-

lem (equilibrium conditions, yield inequality, relations between strain rates and stresses as established by the concept of plastic potential, boundary conditions). It is shown that such stress fields can differ only where both velocity fields furnish vanishing strain rates.

The author then presents generalized forms of two extremum principles which he established earlier [Quart. J. Mech. Appl. Math. 1, 18–28 (1948); J. Appl. Mech. 17, 64–66 (1950); these Rev. 9, 635; 11, 559]. Whereas these principles were originally restricted to continuous velocity fields in the case where the entire body deforms plastically, they are now extended to discontinuous velocity fields which leave part of the body rigid. It is shown how these extremum principles can be used to obtain bounds for the rate of work by the surface tractions on S_V. When the given velocities on S_V are compatible with a rigid body motion (e.g. the prescribed translation of a perfectly smooth or completely rough indenter) the relevant forces or couples associated with this boundary condition are immediately calculable from the rate of work. Finally, when the velocities on S_V are prescribed to be zero and the surface tractions on S_T are given only to within a common factor, the extremum theorems furnish bounds for the magnitude of this factor at the yield point.

As is acknowledged by the author, his treatment of rigid-plastic bodies closely parallels recent American work on limit analysis of elastic-plastic bodies (only part of which seems to have been accessible to the author). With reference to this work he claims that "the relevant theorems are special cases of the maximum work principle and its complement" which he had established in earlier papers, though not in a form sufficiently general for the present purpose. To bolster this erroneous claim he implies that the work of these American authors does not really apply to elastic-plastic materials, and that the definition of a precise moment of yielding in a body made of such a material must be somewhat arbitrary. Actually, the work in question takes full account of the elastic deformations within the precise framework of the linearized theory: the boundary and equilibrium conditions are fulfilled on the undeformed rather than the deformed body. Within this framework, the "yield point" of an elastic-plastic body is sharply defined as that stage of the loading schedule when deformation under constant surface tractions becomes first possible.

W. Prager (Providence, R. I.).

Geiringer, Hilda. Simple waves in the complete general problem of plasticity theory. Proc. Nat. Acad. Sci. U. S. A. 37, 214–220 (1951).

The stress and velocity equations for the plane fully-plastic problem are developed using a general yield condition, the strain rates being given by the concept of plastic potential. The characteristics and characteristic relations are developed for the physical, stress and velocity planes. Initial value problems are discussed for the stress and velocity equations, and explicit solutions obtained in the case of a simple wave or degenerate solution.

E. H. Lee (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Shearer, J. Geometrical optics of concave mirrors and of combinations of mirrors. Australian J. Sci. Research. Ser. A. 3, 532–540 (1950).

The author gives an approximation formula for the astigmatism of spherical and cylindrical mirrors and for correcting astigmatism for two mirror elements with the intention of investigating and correcting the image of objects off axis in a mirror system. The reviewer feels that the author ought to make clearer that he deals only with small surface elements and thus with local correction and not with the mirror as a whole. *M. Herzberger.*

Marx, H. Über die Lange'sche Darstellung der Bildfehler 3. Ordnung. II. Praktischer Teil. Optik 7, 91–95 (1950).

A continuation of an earlier paper [Optik 2, 364–381 (1947); these Rev. 9, 548] in which the author modifies the well-known Seidel aberration formulas by introducing new variables. The new variables are designed to simplify the task of varying the constructional data of a lens so as to reduce the image errors. The second part of the paper discusses difficulties which arise in special cases (in particular when a plane surface is present) and shows how to deal with these difficulties. *M. Herzberger.*

Armsen, Paul. Über die Strahlenbrechung an einer einfachen Sammellinse. II. J. Reine Angew. Math. 188, 65–73 (1950).

[The first part appeared in the same J. 187, 193–221 (1950); these Rev. 12, 222.] The author derives equations for the aberration curves of a "moderately" thick lens.

M. Herzberger (Rochester, N. Y.).

Slevogt, H. Über die Seidelschen Formeln bei Elimination des Objektstrahls. Optik 8, 180–186 (1951).

The Seidel aberrations are given by using two paraxial rays, the rays from the object point and the rays from the center of the central of the entrance pupil. From these one proceeds usually to compute the aberration coefficients which are functions of the object point alone. It is possible however to compute equivalent formulas from which the aperture ray is eliminated and add to the five aberrations as a sixth, for completeness, the spherical aberration of the pupil.

M. Herzberger (Rochester, N. Y.).

Rumsey, N. J. On the extension of a system for differential correction of lens systems to include second-order terms. J. Opt. Soc. Amer. 41, 229–234 (1951).

The possibility is investigated of extending the design procedure developed by Cruickshank for differential correction of lens systems to include the effects of second derivatives. The most convenient variables for use in the first-order work prove unsuitable for the extension to second-order. Choosing new variables, a general method for using second-order coefficients is developed. This threatens to be laborious in practice, but a simplified procedure suitable for the case of bending is described having advantages over other methods commonly used at present. (Author's summary.)

M. Herzberger (Rochester, N. Y.).

Mlodzeevskiĭ, A. B. Deduction of some formulas of geometrical optics. Učenye Zapiski Moskov. Gos. Univ. Fizika. 134, kniga 5, 163–167 (1949). (Russian)

Falla, Louis. Sur la théorie du microscope de phase. Bull. Soc. Roy. Sci. Liège 19, 382–400 (1950).

Teoria elementare per un oggetto costituito da un reticolo di fase. *G. Toraldo di Francia* (Firenze).

Glaser, Walter, und Bergmann, Otto. Über die Tragweite der Begriffe "Brennpunkte" und "Brennweite" in der Elektronenoptik und die starken Elektronenlinsen mit Newtonscher Abbildungsgleichung. II. Z. Angew. Math. Physik 2, 159–188 (1951).

Nella prima parte di questo studio [Z. Angew. Math. Physik 1, 363–379 (1950); questi Rev. 12, 655] gli autori avevano dimostrato che per ogni coppia di punti coniugati esiste una trasformazione parassiale osculatrice di tipo newtoniano, che approssima quella reale fino a termini del quarto ordine. Ora gli autori si dedicano alla ricerca del tipo più generale di campo, per cui la trasformazione osculatrice è la stessa per tutte le coppie di punti coniugati. La condizione necessaria e sufficiente perché ciò avvenga è che, posta l'ascissa lungo l'asse nella forma $z=a\sigma\tan\varphi$ con a e σ opportune costanti, e detto $B_z(z)$ il campo magnetico lungo l'asse, la funzione $B_z^2(a\sigma\tan\varphi)/B_0^2\cos^4\varphi$ risulti periodica col periodo π/ω e $\omega>1$; inoltre bisogna che l'equazione di Hill

$$v''+[1+k^2\sigma^2B_z^2(a\sigma\tan\varphi)/B_0^2\cos^4\varphi]v=0$$

nella quale k è una costante che dipende dalla velocità degli elettroni, ammetta due soluzioni semiperiodiche indipendenti. Vengono dati esempi di campi che soddisfano a queste condizioni. *G. Toraldo di Francia* (Firenze).

Himpan, Joseph. Elektronenoptische Theorie der Ablenkung eines ausgedehnten elektronenoptischen Bildes mittels gekreuzter elektrischer Ablenksysteme. Ann. Physik (6) 8, 405–422 (1951).

L'immagine viene spostata lateralmente dalla sua posizione primitiva per mezzo di due coppie incrociate di placche elettrostatiche. Vengono determinate le aberrazioni del terzo ordine dell'immagine spostata. Queste risultano da prodotti di terzo grado complessivo rispetto allo spostamento, alle coordinate del punto immagine considerato e all'apertura del fascio elettronico. Secondo le denominazioni date dall'autore, si ha: 1) la distorsione senza deformazione, che dipende dallo spostamento dell'immagine, ma non dalle coordinate del punto, né dall'apertura; 2) la distorsione monomorfa, che dipende in secondo grado dallo spostamento e in primo grado dalle coordinate; 3) la distorsione polimorfa, che dipende in primo grado dallo spostamento e in secondo grado dalle coordinate; 4) l'aberrazione monomorfa, che dipende in primo grado dall'apertura e in secondo grado dallo spostamento; 5) l'aberrazione polimorfa, che dipende linearmente dallo spostamento e quadraticamente dall'apertura; 6) la deformazione caustica, che dipende linearmente da ciascuno dei tre gruppi di variabili.
G. Toraldo di Francia (Firenze).

Sturrock, P. A. The aberrations of magnetic electron lenses due to asymmetries. Philos. Trans. Roy. Soc. London. Ser. A. 243, 387–429 (1951).

Viene sviluppato un metodo di perturbazione, considerando le asimmetrie della lente come perturbazioni di una lente ideale. Da prima sono dedotte le formule generali che collegano il campo con la deformazione di una sua superficie equipotenziale. Questa superficie, che è quella delle espansioni polari della lente, viene schematizzata mediante due semicilindri coassiali affacciati, con flange estese all'infinito. La permeabilità magnetica è supposta infinita. Le asimmetrie possono essere di disallineamento (eccentricità e angolo fra gli assi) dei semicilindri o di ellitticità. Trovati i coefficienti del camp perturbato, questi vengono collegati con la funzione caratteristica delle aberrazioni del terzo ordine. Il disallineamento dà luogo con la sua prima potenza a coma e con la seconda potenza ad astigmatismo. L'ellitticità non dà luogo a coma, ma genera astigmatismo proporzionale alla sua prima potenza. L'ellitticità è più nociva quando gli assi maggiori delle ellissi relative ai due cilindri sono incrociati che quando sono paralleli. Vengono calcolati esempi numerici e dedotte tolleranze.
G. Toraldo di Francia (Firenze).

Sturrock, Peter. Propriétés optiques des champs magnétiques de révolution de la forme $H=H_0/[1-(s/\alpha^2)]$ et $H=H_0/[(s/\alpha)^2-1]$ sur l'axe optique. C. R. Acad. Sci. Paris 233, 401–403 (1951).

Tedone, Giuseppe. Sul primo problema della cinematica delle superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 183–187 (1950).

The problem is that of deducing the motion of light wave fronts from given initial conditions and from a given velocity of light as a function of position and direction of propagation. This leads, as is known, to a system of canonical equations. By certain transformations of these equations, the author shows how problems for anisotropic media can sometimes be reduced to corresponding problems for isotropic media. *D. C. Lewis* (Baltimore, Md.).

Severin, Hans. Zur Theorie der Beugung elektromagnetischer Wellen. Z. Physik 129, 426–439 (1951).

Expressions for the electromagnetic field E, H in a point P in the interior of a region bounded by an arbitrary surface are given, provided that the components of E or H have assigned values on the surface. These expressions are

$$4\pi E_P^{(1)}=(ik)^{-1}\operatorname{curl}\iint(df\times H)\Gamma^{(1)}(P,Q),$$

$$4\pi H_P^{(1)}=-\iint(df\times H)\Gamma^{(1)}(P,Q),$$

$$4\pi E_P^{(2)}=-\iint\Gamma^{(2)}(Q,P)(df\times E),$$

$$4\pi H_P^{(2)}=-(ik)^{-1}\operatorname{curl}\iint\Gamma^{(2)}(Q,P)(df\times E),$$

where $\Gamma^{(r)}(P,Q)$ is a tensor with the properties

$$df\times\Gamma^{(1)}(P,Q)=0;\quad df\times\operatorname{curl}\Gamma^{(2)}(P,Q)=0$$

on the surface, and $\Gamma^{(r)}(P,Q)\sim\operatorname{curl}_P r^{-1}e^{-ikr}\delta$ (δ = unit tensor) near the point P. *F. Oberhettinger.*

Shmoys, J. Diffraction of electromagnetic waves by a plane wire grating. J. Opt. Soc. Amer. 41, 324–328 (1951).

The author summarizes his work as follows: Previous work on diffraction by a wire grating is briefly reviewed. The problem of diffraction by a finite grating consisting of perfectly conducting wires of arbitrary cross section is formulated in terms of characteristic plane waves corresponding to the various order spectra defined in optics. Scattering matrix elements are expressed as stationary functionals of current distribution on the grating wires, for the incident wave falling at right angles to the grating elements

and polarized either parallel or perpendicular to them. These are evaluated for the thin wire grating and parallel polarization. *A. E. Heins* (Pittsburgh, Pa.).

Magnus, W., and Oberhettinger, F. On systems of linear equations in the theory of guided waves. Comm. Pure Appl. Math. 3, 393–410 (1950).

In the first part of the paper the authors consider the diffraction of an electromagnetic wave by a plane strip between two parallel planes. In the second part they deal with the diffraction of the dominant transverse electric wave in a rectangular wave guide by a symmetrical strip with its edges parallel to the electric field. Let A_m denote the Fourier coefficients of the current induced in the strip of the first problem [the second problem is similar]. Then, for a strip that covers half of the guide, boundary conditions in the plane of the strip lead to the following system of linear equations:

$$\sum_{m=0}^{\infty} A_m \cos m\zeta = 0, \quad \tfrac{1}{2}\pi < \zeta < \pi,$$

$$\sum_{m=0}^{\infty} A_m (m^2 - c_1)^{\frac{1}{2}} \cos m\zeta = c_2, \quad 0 < \zeta < \tfrac{1}{2}\pi,$$

where c_1 and c_2 are constants. These equations are approximately solved for A_m by application of inversion theorems of Titchmarsh and others. [Reviewer's remarks. The paper is difficult to read; it is not easy to follow the authors as to what they assume, prove, or can prove. In the case of the first problem the authors state that the x and y components of the diffracted wave vanish. In fact, the only vanishing components are E_y, H_x, H_z. In particular, E_x differs from zero, viz.

$$E_x = \mp 60\pi i \sum_{m=0}^{\infty} (\lambda m/b) A_m \sin(2\pi m s/b) \exp(-i|x|k_m), \quad x \gtrless 0.$$

However, this omission does not affect the authors' results. It can be verified that E_x shows the correct singularity at the edge of the strip. The plus sign in Eq. (3.8) seems to be incorrect. For numerical results relating to the second problem, see N. Marcuvitz, Waveguide Handbook, McGraw-Hill, New York, 1951, p. 227.] *C. J. Bouwkamp.*

Vaĭnšteĭn, L. A. The general theory of unsymmetric waves in a circular waveguide with an open end. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 21, 328–345 (1951). (Russian)

L'onda incidente dall'interno della guida è E_{pl} o H_{pl} con $p \neq 0$. Le componenti longitudinali dei vettori di Hertz elettrico e magnetico vengono scritte rispettivamente nella forma

$$\Pi_{ez} = -\frac{2\pi^2 a}{ck} \sin(p\varphi + \varphi_0) \int_C e^{iwz} \begin{Bmatrix} J_p(vr) H_p(va) \\ J_p(va) H_p(vr) \end{Bmatrix} F(w) dw,$$

$$\Pi_{mz} = -\frac{i2\pi^2 a}{c} \cos(p\varphi + \varphi_0) \int_C e^{iwz} \begin{Bmatrix} J_p(vr) H_p'(va) \\ J_p'(va) H_p(vr) \end{Bmatrix} \frac{1}{v} G(w) dw$$

intendendo che le righe superiori valgano all'interno e quelle inferiori all'esterno della guida; r, φ, z sono coordinate cilindriche, a il raggio della guida, $v = (k^2 - v^2)^{\frac{1}{2}}$ e C un opportuno cammino d'integrazione. Se la guida occupa l'intervallo $0 < z < \infty$, la continuità delle componenti tangenziali di H porta le condizioni

$$\int_C e^{iwz} G(w) dw = 0 \quad \text{per} \quad z < 0,$$

$$\int_C e^{iwz} \left[F(w) + \frac{ip}{a} \frac{w}{v^2} G(w) \right] dw = 0 \quad \text{per} \quad z \leq 0,$$

mentre l'annullarsi delle componenti tangenziali di E sulle pareti porta le altre

$$\int_C e^{iwz} v \varphi(wa) F(w) dw = 0 \quad \text{per} \quad z > 0,$$

$$\int_C e^{iwz} \left[\frac{\psi(wa)}{v} G(w) + \frac{ip}{k^2 a} \frac{w \varphi(wa)}{v} F(w) \right] dw = 0 \quad \text{per} \quad z > 0,$$

essendo $\varphi(wa) = \pi va H_p(va) J_p(va)$ e $\psi(wa) = \pi va H_p'(va) J_p'(va)$.

A partire da queste condizioni e da quelle relative a $z = \pm\infty$, l'autore determina per mezzo di considerazioni di teoria delle funzioni il cammino d'integrazione C (costituito dall'asse reale e da un cappio attorno al punto rappresentativo del numero d'onde dell'onda incidente) e la struttura delle funzioni incognite $F(w)$ e $G(w)$. La chiave per la soluzione è la scomposizione di ciascuna delle funzioni φ e ψ nel prodotto di due funzioni olomorfe e senza zeri, l'una nel semipiano complesso superiore, l'altra in quello inferiore. *G. Toraldo di Francia* (Firenze).

Vaĭnšteĭn, L. A. Numerical results of the theory of unsymmetric waves in a circular waveguide with an open end (the waves E_1 and H_1). Akad. Nauk SSSR. Žurnal Tehn. Fiz. 21, 346–357 (1951). (Russian)

I risultati teorici del lavoro precedente vengono utilizzati per il calcolo numerico nel caso delle onde E_1 e H_1. È rappresentato graficamente l'andamento dei coefficienti di riflessione e di trasformazione dell'onda incidente nei vari modi della guida. Sono studiate le caratteristiche d'irraggiamento e le sezioni equivalenti di assorbimento nei vari casi. Viene effettuato un confronto fra i risultati della teoria rigorosa e quelli dedotti mediante il principio di Huygens. *G. Toraldo di Francia* (Firenze).

Vaĭnšteĭn, L. On the theory of forced oscillations of a two-wire line. Učenye Zapiski Moskov. Gos. Univ. Fizika. 134, kniga 5, 24–30 (1949). (Russian)

Twiss, R. Q. On Bailey's theory of amplified circularly polarized waves in an ionized medium. Physical Rev. (2) 84, 448–457 (1951).

The author analyses carefully Bailey's theory of propagation of waves in an ionized medium. In his analysis he considers only the transverse fields and ignores the longitudinal oscillations. A plane electromagnetic wave is normally incident on the interface between two media, of which the upper one contains a uniform electron stream with infinitely massive positive ions at a constant velocity under the action of an external constant magnetic field. The free charges in the medium I are assumed to be stationary. It is then shown that the growing waves arrived at by Bailey can only be excited by reflection and that hence his conclusions as to sunspot and tube noise simulated by such waves is in error. *M. J. O. Strutt* (Zurich).

Twersky, Victor. Multiple scattering of radiation. I. Arbitrary configuration of parallel cylinders, planar configurations, two cylinders. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. EM-34, ii+68 pp. (1951).

Starting from the scattering of plane waves by a single cylinder, a solution is obtained for the scattering of waves by an arbitrary configuration of parallel cylinders. This solution is expressed as the incident wave plus the sum of an infinite number of orders of scattering. It is shown that for a planar configuration, provided the spacing of the cylinders

is large compared to the wave-length, the multiple scattered contributions are symmetrical with respect to the plane of the cylinders' axes. The problem of the two cylinders and of two bosses is considered in detail. *M. J. O. Strutt.*

Poritsky, H., and Weil, H. Conduction of current in a metallic pipe filled with a conducting liquid. J. Appl. Phys. **22**, 1002–1005 (1951).

A metallic pipe considered as a cylindrical shell of thickness t and conductivity i_s is filled partly or completely with a liquid of conductivity i_u. An electric current I is allowed to enter and leave at two diametrically opposite points on the outside surface of the cylindrical shell of radius b. The problem is the complete evaluation of the potential distribution. This is done by means of the Fourier integral development satisfying on the outer surface the boundary condition of vanishing radial flow except at the electrode points where the current density is expressed as the product of δ-functions in angle and in axial coordinate z. The Fourier integral for the potential variation on the surface $r=b$ is then evaluated by residues. For practical computations, several approximate solutions are given by considering the pipe straightened out into a flat plate and applying the method of images, representing the current source as a point source within a plate of twice the thickness of the cylindrical shell. The special cases of vanishing conductivity of the liquid and of a thin film of the liquid are treated briefly.

E. Weber (Brooklyn, N. Y.).

Breus, K. A. The potential field of a charged sphere with two openings. Ukrain. Mat. Žurnal **2**, no. 1, 86–106 (1950). (Russian)

The problem described in the title leads to Lamé functions. By an inversion, the spherical belt can be converted into a flat ring, and the potential problem of the flat ring has been discussed by E. G. C. Poole [Proc. London Math. Soc. (2) **29**, 342–354 (1929); **30**, 174–186 (1929)] and Chester Snow [The hypergeometric and Legendre functions . . . , NBS Mathematical Tables, 1942; these Rev. **4**, 197]. In the present paper the author discusses the problem without using the theory of the Lamé equation or any previous work on the subject. *A. Erdélyi* (Pasadena, Calif.).

Berger, Erich R. Zum zweidimensionalen Feldproblem zweier leitender Ebenen. Österreich. Ing.-Arch. **5**, 174–182 (1951).

The two-dimensional field between symmetrically placed oppositely charged plates of the same width is calculated by the use of the Schwarz-Christoffel formula and a numerical example is given. The capacity of the system is also computed. *C. Saltzer* (Cleveland, Ohio).

Lundquist, S. On the stability of magneto-hydrostatic fields. Physical Rev. (2) **83**, 307–311 (1951).

The stability of magneto-hydrostatic fields for static deformations is considered. For a medium with infinite conductivity, a deformation $\xi(\mathbf{r})$ carries the magnetic field $\mathbf{H}$ at $\mathbf{r}$ into $\mathbf{H}+(\mathbf{H}\cdot\text{grad})\,\xi$ at $\mathbf{r}+\xi$. The change in magnetic energy caused by the displacement ξ is evaluated and expressed in the form

$$\Delta W=\frac{1}{2}\int\left(\psi\frac{\partial\xi_i}{\partial x_j}\frac{\partial\xi_j}{\partial x_i}+H_iH_k\frac{\partial\xi_j}{\partial x_i}\frac{\partial\xi_j}{\partial x_k}\right)d\tau,$$

where $\psi=(p+\frac{1}{2}|\mathbf{H}|^2)/4\pi$ (p denotes the pressure) and the integral is extended over the volume considered. It is assumed that the field is unstable if ΔW is negative. By considering a special type of deformation of an axially symmetrical field $\mathbf{H}=[-H_\varphi(r)y/r, H_\varphi(r)x/r, H_z(r)]$, the author concludes that instability will arise when $(H_\varphi^2)_{av}>2(H_z^2)_{av}$ i.e., when the "magnetic energy due to the twisting exceeds double the energy of the non-twisted field".

S. Chandrasekhar (Williams Bay, Wis.).

Synge, J. L. The fundamental theorem of electrical networks. Quart. Appl. Math. **9**, 113–127 (1951).

The fundamental problem of electrical networks is to find the currents i flowing in the branches when the voltages e of the generators in the branches are given. A neat solution of this problem is given by Kron's formula, $i=C(C_tZC)^{-1}C_te$. Here C is the incidence matrix between branches and a basic set of meshes; C_t is the transpose of C. The matrix Z is determined by the type of electromagnetic coupling between branches; for the purpose of this paper it may be regarded as an arbitrary matrix. This paper gives a proof of Kron's formula. The proof starts from first principles and carefully distinguishes the topological, algebraic, and physical notions involved. The details are carried out so as to be understandable by engineers unfamiliar with topology or by topologists unfamiliar with engineering. *R. J. Duffin.*

Gilbert, E. N. *N*-terminal switching circuits. Bell System Tech. J. **30**, 668–688 (1951).

A black box has N terminals and M shafts. Each shaft turns one or more selector switches inside the box. Each switch consists of an arm attached to the shaft and several stationary contacts. The particular contact touched by the arm depends on the angular position of the shaft. The terminals, arms, and contacts are connected in some way inside the box. The switching function is the pattern of interconnection of the terminals determined by the settings of the shafts. A method is given for synthesizing such a switching function. This method is shown to be economical in the sense that the switching functions which can be synthesized by any other method using much fewer contacts comprise a vanishingly small fraction of the total of all possible switching functions. If each shaft has only two positions the problem reduces to one already considered by C. E. Shannon [Bell System Tech. J. **28**, 59–98 (1949); these Rev. **10**, 671]. *R. J. Duffin* (Pittsburgh, Pa.).

Zimmermann, F. Über Eigenschaften der Transformatorschaltgruppen in Matrizendarstellung. Österreich. Ing.-Arch. **5**, 105–116 (1951).

The currents in the primary and the secondary windings of a threephase transformer are connected by a linear transformation. The three by three matrices of such transformations are tabulated for various types of circuit arrangements.

R. J. Duffin (Pittsburgh, Pa.).

Schulz, Hermann. Die Transformation der Vierpol-Kettenmatrix in die Diagonalform. Eine Systematik aller linearen Vierpole und ihre Schaltungs-Symbolik. Arch. Elektr. Übertragung **5**, 257–266 (1951).

This paper analyzes the two by two matrix A which transforms the voltage and current leaving the output terminals of a four-pole network, into the voltage and current entering the input terminals. Multiplication of such matrices is interpreted as connecting the corresponding networks in a chain. It is shown how to write A as a product of

four factors. This leads to an equivalent circuit for the network as a chain of four simpler networks.
R. J. Duffin (Pittsburgh, Pa.).

Fialkow, Aaron, and Gerst, Irving. The transfer function of an R-C ladder network. J. Math. Physics 30, 49–72 (1951).

The ratio of output to input voltage of a four-terminal network is termed the transfer function A. It is a rational function of the complex frequency p,

$$A(p)=K\prod(p+\delta_i)/\prod(p+\gamma_i).$$

The authors first consider an L network, that is, two two-terminal networks in series, the output being across one of them. The two-terminal networks are permitted to be arbitrary networks of resistors and capacitors. Making use of the well known properties of such two-terminal networks they deduce: (a) The γ_i are positive and distinct. (b) The δ_i are positive or zero. Let $s_1\leqq s_2\leqq\cdots\leqq s_r$ be the sequence of the γ_i and δ_i together, so r is the sum of the degrees of the numerator and denominator of A. Then each pair (s_{2k-1}, s_{2k}) is of the form (δ, γ) or (γ, δ) and if r is odd then s_r is a γ. (c) The range of K is an interval starting from zero and going up to a positive value K_0, beyond which the equation $\prod(p+\gamma_i)-K\prod(p+\delta_i)=0$ for the first time fails to have all negative or zero roots. They also give a more explicit determination of K_0. This evaluation of the range of K is one of the novel results of the paper. Given a rational function A which satisfies (a), (b), and (c) they show how to synthesize an L network which has A as its transfer ratio; so these conditions are both necessary and sufficient. They obtain almost as precise a characterization for a ladder network, a chain of L networks.
R. J. Duffin.

***Schultze, Ernst. Über einige Approximationen, die bei der Synthese elektrischer Netzwerke mit vorgegebenen Eigenschaften nötig sind.** Thesis, Eidgenössische Technische Hochschule in Zürich, 1951. 68 pp.

Given a filter with a certain ideal characteristic the problem considered is the design of a network with a finite number of elements which will approximate the ideal. The criterion of approximation is taken to be the mean square deviation relative to certain weight functions. Under suitable conditions this gives pointwise approximation also. One problem considered in detail concerns a filter giving constant attenuation to low frequency signals and constant phase shift to high frequency signals. Such an ideal frequency characteristic is taken to be $((p^2+1)^{\frac{1}{2}}-p)^{\frac{1}{2}}$ where p is the complex frequency. An approximation is found as a sixth degree polynomial in $(b-p)/(b+p)$ where b is a constant. This rational function is then synthesized by the method of Brune as a two-pole network with fourteen elements. The proof of the approximation is made to depend on theorems of J. L. Walsh [Interpolation and Approximation ..., Amer. Math. Soc. Colloq. Publ., v. 20, New York, 1935]. Another problem concerns a filter which will delay an arbitrary signal by one second. This problem requires a suitable approximation of the Dirac function $\delta(t-1)$ for $0\leqq t<\infty$. The approximation is carried out as a finite series of Laguerre functions. This leads, therefore, to a Lee-Wiener four-pole network. The proof of the approximation employs properties of Laguerre polynomials given by G. N. Watson, G. Szegö, E. Moecklin, and F. Tricomi.
R. J. Duffin.

Kraus, G. Ein Umkehrungssatz in nichtlinearen Wechselstromschaltungen. Österreich. Ing.-Arch. 5, 48–73 (1951).

Circuits constructed with elements characterized by a non-linear relation between voltage and current are discussed. Such elements are assumed to be frequency independent with current-voltage characteristics which are single-valued, continuous, and continuously differentiable. A reciprocity theorem, analogous to Kirchhoff's theorem for linear networks, is derived for several cases of two-pole and four-pole networks. This theorem is expressed in terms of certain relationships between partial derivatives of applied voltages and currents.
R. Kahal (St. Louis, Mo.).

Goldberger de Buda, R. Zur Frage der Entzerrung eines Impulsverstärkers. Österreich. Ing.-Arch. 5, 74–80 (1951).

The conversion of an ordinary amplifier into a pulse amplifier for a fixed repetition frequency is treated. The input and output signals are approximated by the sum of their first N harmonics, and a driving point impedance for insertion in the amplifier is chosen so as to minimize a distortion factor defined in terms of the harmonic coefficients. The problem is then reduced to that of determining a network whose driving point impedance is prescribed for N frequencies.
R. Kahal (St. Louis, Mo.).

Quantum Mechanics

Cade, R. Curvilinear momenta in quantum mechanics. Proc. Cambridge Philos. Soc. 47, 451–453 (1951).

The author shows that the real quantum mechanical momenta p_r, corresponding to a set of general curvilinear coordinates $q_r=q_r(x_1, x_2, \cdots, x_n)$ are uniquely defined as functions of the Cartesian coordinates $x_1, \cdots, x_n$ and momenta by the commutation relations

$$[q_r, p_s]=i\hbar\delta_{rs}, \quad [p_r, p_s]=0$$

except for arbitrary phase functions and provided that the q_r satisfy a certain general condition which usually obtains in practice. Consistent with the assumption that the x_r can have as eigenvalues all real numbers, the eigenvalues of the q_r are all the real numbers in the intervals in which the variables are defined. The author shows that the properties of eigenvalues of the p_r are fixed by those of the q_r and by the definitions of the p_r.
M. Pinl (Dacca).

Viguier, Gabriel. Enchainement et quantitation. Cas du rotateur sphérique. Bull. Soc. Math. Phys. Macédoine 2, 81–89 (1951).

Verf. geht von Schrödingers Wellengleichung im Falle einer Partikel bestimmter Ladung aus und betrachtet monochromatische Lösungen und die Transformation

$$Y_k=\psi\exp[-2\pi iE_kt/h]$$

der Wellenfunktion ψ. Die Funktionen $Y_k(\vartheta, \varphi)$ sind dann Laplacesche Kugelfunktionen. Für $x=\cos\vartheta$ und $Y_k=e^{\pm im\varphi}P_k{}^m(x)$ ergibt sich im Falle des räumlichen Rotators die gewöhnliche Differentialgleichung

$$(*)\quad \sin^2\vartheta\, d^2P_k(x)/dx^2-2\cos\vartheta\, dP_k(x)/dx+k(k+1)P_k(x)=0$$

fur die Legendreschen Polynome $P_k^m(\cos\vartheta)$ und die Energieeigenwerte $E_k=k(k+1)h^2/8\pi^2I$ (h Plancksches Wirkungsquantum, I Trägheitsmoment des Rotators). Gleichwie nun

im Falle des linearen Planckschen Oszillators wichtige Eigenschaften der Hermiteschen Polynome mit solchen Riccatischer Differentialgleichungen in Verbindung gebracht werden können [vgl. Viguier, Ann. Fac. Sci. Univ. Toulouse (4) 9 (1945), 1–64 (1948); diese Rev. 11, 176], gelingt dies nunmehr auch Verf. im Falle des räumlichen Rotators mit Legendre'schen Polynomen. Die zugeordnete Riccatische Differentialgleichung lässt sich durch weitere Transformation auf die Gestalt

$$d\tau/dx+\tau^2/(1-x^2)+k(k+1)=0$$

bringen. Eine Ähnlichkeitstransformation führt dann auf $d\gamma/dx+\gamma^2=k(k+1)/(x^2-1)$ und die Substitution $\gamma=v_0'/v_0$ schliesslich auf die Gleichung zweiter Ordnung

$$v_0''/v_0=k(k+x^2)/(x^2-1)-k,$$

welche die Anwendung des sogenannten Kettenverfahrens von G. Darboux [Théorie générale des surfaces . . . , tomes 1 et 2, Gauthier-Villars, Paris, 1887 et 1889] gestattet. Damit gewinnt Verf. gewisse Analogien zwischen Quantelung und Darbouxschem Kettenverfahren, wie A. Buhl in solchen Fallen schon vermutet hatte. *M. Pinl* (Dacca).

Stueckelberg, E. C. G. Relativistic quantum theory for finite time intervals. Physical Rev. (2) 81, 130–133 (1951).

Let V be the 4-dimensional space-time volume lying between two times t' and t''. Let $V(x)$ be the function equal to unity inside and to zero outside V. Consider a transition process in relativistic quantum theory, in which particles with known wave-functions $u', v', \cdots$ at the time t' become scattered or transmuted into a set of particles with wave-functions $u'', v'', \cdots$ at time t''. The nth order contribution to the transition matrix element for this process is

$$(1)\quad S_n(V)=\int dx_1\cdots\int dx_n V(x_1)\cdots V(x_n) \times[\bar{u}''(x_1)\bar{v}''(x_2)\cdots u'(x_m)v'(x_{m+1})\cdots]\Delta_n^c.$$

Here the integration is over n space-time points $x_1, \cdots, x_n$, and Δ_n^c is a function of the coordinate differences (x_i-x_j) only. In Stueckelberg's formulation of relativistic quantum theory, equation (1) is taken as the fundamental postulate. The form of the function Δ_n^c is restricted only by the two requirements of unitarity and causality. Unitarity means that the transition matrix S taken as a whole is unitary. Causality means that particles interact only by their retarded fields, so that Δ_n^c is built up of a network of factors $\Delta^c(x_i-x_j)$, where Δ^c is the special causal function of Stueckelberg [Stueckelberg and Rivier, Physical Rev. (2) 74, 218 (1948)]. Stueckelberg has proved that these two requirements determine the function Δ_n^c uniquely, apart from a small number of arbitrary additive constants, if the form of the first-order transition matrix $S_1(V)$ is given [Stueckelberg and Rivier, Helvetica Phys. Acta 23, 215–222, 236–239 (1950); these Rev. 11, 569, 763]. Thus the postulate (1), in spite of its great generality, is in fact a sufficient basis on which to build up the whole of relativistic quantum theory.

In the integrals (1) there appear divergences due to the sharp discontinuity of the integrand at the boundary of V. It is therefore necessary, in order to make a consistent theory, to work with integrals (1) in which $V(x)$ is a continuous function. In particular, the author considers the case in which $V(x)$ denotes the probability that the point x lies inside the volume V, when the beginning and end of V are not fixed times but are random variables with the probability distributions $f'(t)$ and $f''(t)$ respectively. If f' and f'' are continuous functions, then $V(x)$ is continuous and the boundary divergences of (1) disappear. It is supposed that f' and f'' represent distributions around the times t' and t'' with small widths $\Delta t'$ and $\Delta t''$. Then (1) gives the probability amplitude for a transition, in which the initial state $u', v', \cdots$ is observed at approximately the time t', and the final state $u'', v'', \cdots$ is observed at approximately the time t'', the time-measurements being uncertain by $\Delta t'$, $\Delta t''$ respectively. All physically observed transitions are observed with finite uncertainties in the time measurements, because only a finite amount of energy can be exchanged between the system and the measuring apparatus. Therefore it is reasonable that a divergence-free and directly interpretable theory should be obtained only by using functions $V(x)$ of the kind here described.

In the last section the method is illustrated by a calculation of the radiation emitted by a free electron during a finite time interval. The result is in agreement with the usual theory. *F. J. Dyson* (Ithaca, N. Y.).

Stueckelberg, E. C. G., et Green, T. A. Élimination des constantes arbitraires dans la théorie relativiste des quanta. Helvetica Phys. Acta 24, 153–174 (1951).

The first part of this paper describes in detail the physical model which Stueckelberg has introduced [see the preceding review] to deal with localized events in relativistic quantum theory. Using this model, probability amplitudes are expressed as integrals containing two types of terms; terms of the first type represent processes occurring with conservation of energy and momentum, and are proportional to the four-dimensional volume V in which the processes occur; terms of the second type are boundary effects, arising from the interaction of the observed system with the measuring apparatus which is used to detect particles entering or leaving the region, and represent nonconservative processes. By making the volume V tend to infinity a clear separation of the two types of terms is achieved.

The second part of the paper contains the application of the method to the study of all processes occurring in quantum electrodynamics in the second, third and fourth order in the coupling constant e. It is shown that a particular kind of divergent effect known as "wave-function renormalization", which introduces a troublesome ambiguity into the calculations of Schwinger [Physical Rev. (2) 76, 790–817 (1949); these Rev. 11, 569] and Dyson [ibid. 75, 1736–1755 (1949); these Rev. 11, 145], is here removed unambiguously by the correct explicit treatment of boundary effects. The remaining ambiguities in the Stueckelberg theory are the arbitrary constants which appear in the causal functions Δ_n^c, replacing the divergent mass and charge renormalization factors which occur in the Schwinger theory. The authors show here that these arbitrary constants can also in the Stueckelberg theory be consistently interpreted as mass and and charge renormalization effects, at least up to the order e^4. Therefore, after the renormalizations have been carried through, the theory gives finite and unambiguous expressions for all transition matrix elements.

The importance of this conclusion lies in the fact that Stueckelberg develops his theory from basic postulates which are apparently much less restrictive than the Hamiltonian formalism used by Schwinger. It is thus proved that any theory satisfying general requirements of unitarity and causality, and agreeing with quantum electrodynamics in the lowest approximation, will give results everywhere identical with those of Schwinger. *F. J. Dyson* (Ithaca, N. Y.).

Hayakawa, Satio, Miyamoto, Yonezi, and Tomonaga, Sin-itiro. On the elimination of the auxiliary condition in the quantum electrodynamics. I. J. Phys. Soc. Japan **2**, 172–183 (1947).

The first satisfactory quantum-theoretical treatment of electrons interacting with the electromagnetic field was the many-time theory of Dirac [Dirac, Fock, and Podolsky, Phys. Z. Sowjetunion **2**, 468–479 (1932)]. This theory was perfectly relativistic in form. However, in order to derive from it the actual interaction between electrons, and to eliminate the auxiliary condition ($\partial A_\mu/\partial x_\mu = 0$), it has been customary to make a noncovariant separation between the longitudinal and transverse components of the field. The object of this paper is to carry through the elimination of the auxiliary condition, and to determine the interaction between electrons, preserving at all stages the formal covariance of the theory. Since the paper was written, the Dirac many-time theory has been made obsolete by the simpler super-many-time theory [cf. e.g. Tomonaga, Progress Theoret. Physics **1**, 27–42 (1946); Schwinger, Physical Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 226, 345]. However, the covariant elimination of the auxiliary condition, as here described, applies almost without change in the newer formalism. *F. J. Dyson* (Ithaca, N. Y.).

Hayakawa, Satio, Miyamoto, Yonezi, and Tomonaga, Sin-itiro. On the elimination of the auxiliary condition in the quantum electrodynamics. II. J. Phys. Soc. Japan **2**, 199–204 (1947).

The methods of part I of this paper [see the preceding review] are here applied to two special problems: the self-energy of an electron, and the Bremsstrahlung emitted by an electron in consequence of a sudden change of velocity. Results are in agreement with those found earlier by other methods. *F. J. Dyson* (Ithaca, N. Y.).

Rohrlich, F. Quantum electrodynamics of charged particles without spin. Physical Rev. (2) **80**, 666–687 (1950).

Equivalence of the Feynman theory and Kanesawa and Tomonaga's [Progress Theoret. Physics **3**, 1–13 (1948); these Rev. **10**, 227] extension of the Schwinger-Tomonaga theory is shown for the interaction of positively and negatively charged spinless mesons with the electromagnetic field. The identity of the scattering matrix for the two theories is shown: its divergencies are investigated and removed by renormalization. The interaction Hamiltonian is nonlinear in the electromagnetic potentials and gives rise to double corners in the Feynman diagrams. One of the divergencies is associated with the meson-meson interaction and is removed by the introduction of a direct interaction term in the Hamiltonian. This renormalization leaves a finite interaction which must be determined from experiment. The cancellation of certain divergencies for particles of spin 0, ½ and 1 is shown in an appendix. *C. Strachan.*

Rayski, J. On the reciprocal field theory. Proc. Phys. Soc. Sect. A. **64**, 657–658 (1951).

A nonlocal complex scalar field is in interaction with a real local scalar field. It is known that the field equations cannot be differential. An integral equation for the fields and a generalization of the Δ-functions to nonlocal fields are proposed. The author concludes "we possess a complete quantum formalism for the nonlocal scalar fields including interaction with full reciprocal symmetry. The formalism is free of the usual convergence difficulties." *A. J. Coleman.*

Yukawa, Jiro, and Umezawa, Hiroomi. On the problem of covariance in quantum electrodynamics. I. Progress Theoret. Physics **6**, 112–121 (1951).

A formula of Pais and Epstein [Rev. Modern Physics **21**, 445–446 (1949)] for the self-stress of the electron, $\int T_{11}(0)dv(0) = -(e^2/2\pi\hbar c)\mu c^2$ where μ is the inertial mass of the electron, is confirmed by a more explicit derivation. It appears that it is not possible to avoid the appearance of such a self-stress of "mass type" in standard quantum-electrodynamics. The authors show, however, that in C-meson theory this self-stress just cancels out with a finite C-mesonic self-stress of the same type. In an addendum, the additional self-stress due to vacuum polarization is discussed. *E. Gora* (Providence, R. I.).

Yukawa, Jiro, and Umezawa, Hiroomi. On the problem of covariance in quantum electrodynamics. II. Progress Theoret. Physics **6**, 197–201 (1951).

It is shown that a non-vanishing self-energy of the photon implies also a non-vanishing self-stress. Both effects are due to vacuum polarization. *E. Gora* (Providence, R. I.).

Valatin, J. G. On a formulation of quantum electrodynamics. Physical Rev. (2) **83**, 850–851 (1951).

Ivanenko, D., and Grigor'ev, V. On an interpretation of regularization in quantum electrodynamics. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 563–566 (1951). (Russian)

The authors' point of view is that it should be possible to interpret physically the mathematical devices of regularization procedures, and that it should not be necessary to distinguish between "realistic" and "formalistic" theories. In particular, Feynman's "formalistic" use of an auxiliary mass [Physical Rev. (2) **74**, 1430–1438 (1948); these Rev. **10**, 345] is shown to be equivalent to a replacement of the ordinary field equations by higher order equations. A fifth coordinate is introduced in these equations, and interpreted as an internal degree of freedom; it is linked up with the relation between the rest-mass of the field and the constant which determines the interaction between field and particles. The well-known conditions for cancellation of the singularities [Pauli and Villars, Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] follow from these higher order equations if a discrete spectrum of auxiliary masses is assumed. It is further shown that such a regularization procedure does not remove the infinite energy of the zero-point fluctuations. *E. Gora* (Providence, R. I.).

Steinwedel, Helmut. Zur Strahlungsrückwirkung in der klassischen Mesonentheorie. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950**, 275–280 (1950).

The right-hand side of the equation of motion

$$\tfrac{2}{3}g^2c^{-3}d^3\mathbf{r}/dt^3 = \tfrac{1}{2}g(\mathbf{E}_{\mathrm{ret}} - \mathbf{E}_{\mathrm{adv}})$$

analogous to the Dirac [Proc. Roy. Soc. London Ser. A. **167**, 148–169 (1938)] and Wheeler and Feynman [Rev. Modern Physics **17**, 157–181 (1945); **21**, 425–433 (1949); these Rev. **11**, 293] equation for the motion of a point electric charge influenced by its own electromagnetic radiation, is evaluated where g is the mesic charge of the moving nucleon (only periodic motion is considered). The meson field vectors $\mathbf{E}_{\mathrm{adv}}$, $\mathbf{E}_{\mathrm{ret}}$ are derived from advanced and retarded potentials of the vector meson field. *C. Strachan* (Aberdeen).

Steinwedel, Helmut. Die klassische Mesodynamik als Fernwirkungstheorie. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950**, 281–286 (1950).

Following a brief summary of the action-at-a-distance electrodynamics of Wheeler and Feynman [Rev. Modern Physics **17**, 157–181 (1945); **21**, 425–433 (1949); these Rev. **11**, 293] and of the use of the Schwartzschild-Tetrode-Fokker variation principle [e.g., Z. Physik **58**, 386–393 (1929)], classical vector meson theory is similarly formulated. The complicated resulting equations of motion are given for the non-relativistic, simple-harmonic vibration of a particle. *C. Strachan* (Aberdeen).

Steinwedel, Helmut. Zur Strahlungsrückwirkung in der Wheeler-Feynmanschen Neuformulierung der Elektronentheorie. Z. Naturforschung **6a**, 173–177 (1951).

Methods developed in the author's [same vol. 123–133 (1951); these Rev. **12**, 888] generalized linear field theory where the delta-function is replaced by a non-singular function $f(\Gamma)$ ($\Gamma=-(a_\nu-b_\nu)(a^\nu-b^\nu)$, a_μ, b_μ space-time coordinates), are applied to Feynman's formulation of classical electron theory [Physical Rev. (2) **74**, 939–946 (1948); these Rev. **10**, 222]. This is done in such a way that all but the retarded fields disappear in the equations of motion. In addition to the usual mass and radiation reaction terms these equations contain terms with higher order time-derivatives which depend upon the exact form of $f(\Gamma)$. The usual argument that linear equations of the customary type do not possess stable solutions does not apply to these higher order equations, but, at the same time, no proof is given that these equations will always possess stable solutions. The particular form of $f(\Gamma)$ which corresponds to Feynman's theory is determined explicitly. The possibility of analogous generalizations in quantum electrodynamics is discussed briefly. *E. Gora* (Providence, R. I.).

Belinfante, F. J. "Integro-causality" in convergent quantum theory of fields. Progress Theoret. Physics **6**, 202–206 (1951). (Esperanto. English summary)

The standard subtraction procedures of quantum electrodynamics are not unambiguously defined for processes which take place between finite time limits t_0 and t. To overcome this difficulty the author proposes to replace the Schrödinger equation by the "integro-causal" relation $i\hbar\partial\Psi_{t_0}(t)/\partial t=B_{t_0}(t)\Psi_{t_0}(t)$ where the interaction operator $B_{t_0}(t)$ has a "memory" and no longer leads to an "immediate" causal relation between $\Psi(t)$ and $\Psi(t+dt)$. In such a theory it is not permissible simply to omit the parts of the Feynman graphs which lead to divergencies. A modification of this procedure is proposed which takes the finite time limits into account. *E. Gora* (Providence, R. I.).

Ôno, Yôrô, and Sugawara, Masao. Behavior of D-function in Yukawa's non-local field theory. Progress Theoret. Physics **6**, 182–187 (1951).

The Δ-function of localized field theories (i.e., the commutator of non-interacting field operators at different space-time points) is generalized to Yukawa's non-local field theory [H. Yukawa, Physical Rev. (2) **77**, 219–226 (1950); these Rev. **11**, 567]. This commutator is shown to vanish unless the "relative" coordinates of the two operators are antiparallel. In this case it is a function of the difference of the "mean" coordinates which is singular on a wedge with axis in the direction of the "relative" coordinates instead of on the customary light cone. The time derivative at zero behaves just as the local Δ function with respect to directions perpendicular to the relative coordinate, vanishing except in the immediate vicinity of zero. However, it is non-vanishing (indeed infinite) for points arbitrarily far from the origin in the direction of the "relative" coordinate. *K. M. Case* (Ann Arbor, Mich.).

Leite Lopes, J. On the particle picture of quantized Bose fields. I. Anais Acad. Brasil. Ci. **23**, 39–60 (1951).

This paper generalizes to relativistic particles the work of Fock [Z. Physik **75**, 622–647 (1932)] showing the equivalence between the treatment of many particle problems by second quantization and by the configuration space method. First it is shown how the field operators corresponding to a neutral, scalar, boson field can be expressed as operators in a linear space which is a direct product of an enumerable set of Hilbert spaces. Though the wave functions in this space are complex it is proved that the electric current vanishes identically. In the second part of the paper it is shown how a similar representation may be found for the non-hermitian operators of a charged scalar boson field. Later applications to vector meson and photon fields are promised. *K. M. Case* (Ann Arbor, Mich.).

Schönberg, M. Sur la théorie des perturbations en mécanique quantique. I. Spectres discontinus. Nuovo Cimento (9) **8**, 243–270 (1951).

Standard perturbation methods are modified. The modification of Dirac's time-dependent theory contains no secular terms and is directly related to a modification of Schrödinger's time-independent theory. The expansions are not in powers of a parameter but are obtained like a Liouville-Neumann series and they contain the perturbed eigenvalues; their convergence is not studied. The Heisenberg and density operators are also considered. *T. E. Hull* (Vancouver, B. C.).

Schönberg, M. Sur la théorie des perturbations en mécanique quantique. II. Spectres continus et mixtes. Nuovo Cimento (9) **8**, 403–431 (1951).

Methods given in I [see preceding review] are adapted to continuous and mixed spectra. Systems of particles and quantum fields are considered. The self-energy of a state in the continuous spectrum is not determined by the perturbation calculations because of the ambiguity in the correspondence between perturbed and unperturbed levels. A modified form of the Heitler-Peng equation is given. The secular terms can be eliminated from the expansion of the unitary operator for the motion. *T. E. Hull.*

Nishijima, Kazuhiko. Note on the eigenvalue problem in the quantum field theory. Progress Theoret. Physics **6**, 37–47 (1951).

In quantum field theory it is customary to introduce an operator U, defined as a formal power-series in the interaction, which transforms the total energy of a system with interaction into an operator which is diagonal in the representation in which the energy of free non-interacting fields is diagonal. By means of this operator U the determination of the properties of interacting fields is formally reduced to a problem involving only non-interacting fields. The purpose of this paper is to point out that the reduction of the problem is in many circumstances illusory, since U is not a unitary operator and the series by which U is defined is not convergent. In particular, if U were a well-defined unitary operator the absurd conclusion would be reached that the interacting fields have identically the same energy eigen-

values as non-interacting fields. The conclusions of this paper are not new but are here for the first time concisely stated. *F. J. Dyson* (Ithaca, N. Y.).

Bogolyubov, N. N. On a new form of the adiabatic theory of disturbances in the problem of interaction of particles with a quantum field. Ukrain. Mat. Žurnal 2, no. 2, 3–24 (1950). (Russian)

A particle of position r and momentum p is in interaction with a field specified by operators q_f where f is a three-dimensional vector which can take on a discrete or continuous infinity of values. The hamiltonian for the system has the form

$$H=\frac{1}{2\mu}p^2+\sum_f \alpha_f q_f e^{if\cdot r}+\tfrac{1}{2}\sum_f v_f q_{-f} q_f+\frac{\epsilon^4}{2}\sum_f v_f p_{-f} p_f,$$

where $p_f=-i\partial/\partial q_f$, μ, v_f and ϵ are real and α_f are complex numbers. Such an H occurs in Pekar's theory of an electron in an ionic lattice [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 19, 796–806 (1949)]. The author defines the "adiabatic" case as that in which the last term in H (the "kinetic energy of the field") is small compared with the other terms, so ϵ is a small parameter. The fact that H is invariant under the transformation $r\to r+a$, $q_f\to q_f e^{-if\cdot a}$ motivates the introduction of new variables and in terms of these H is developed in powers of ϵ: $H=H_0+\epsilon H_1+\epsilon^2 H_2+\cdots$. Explicit expressions are given for H_0, H_1 and H_2. The position r is split into two parts $r=q+\lambda$, where q, which gives the mean position, changes uniformily and λ gives the fluctuation. Assuming that the first approximation to the wave function separates into a product of $\varphi(\lambda)$ depending only on λ and a factor depending on the new field variables, a first approximation E_0 to the energy of the system is obtained and hence an equivalent effective mass μ_{eff} is deduced. (It seems that there is a redundant 3 in the expression for μ_{eff}). Using semiclassical theory in which the field was not quantized, Pekar [ibid. 16, 341–347 (1946)] had previously obtained the same expression for E_0 and Landau and Pekar [ibid. 18, 419–423 (1948)] the formula for μ_{eff}. The author goes a step further, and using his expression for H_2 obtains a second approximation to the energy and the effective mass valid up to terms quadratic in ϵ. *A. J. Coleman.*

Valatin, Jean G. Sur la seconde quantification. II. Théorie du positron. J. Phys. Radium (8) 12, 542–549 (1951).

The decomposition of the space of wave functions for a free relativistic particle into manifolds on which the energy operator is respectively positive and negative induces corresponding decompositions for operators on the corresponding space for an arbitrary number of such particles (with Fermi statistics). The latter decompositions are developed in terms of the author's formalism [same J. (8) 12, 131–141 (1951); these Rev. 12, 784]. *I. E. Segal* (Princeton, N. J.).

Kwal, Bernard, et de Broglie, Louis. Quelques considérations sur les transformations de jauge et la définition des tenseurs de Hertz en théorie du corpuscule maxwellien de spin 1. C. R. Acad. Sci. Paris 232, 2056–2058 (1951).

The authors show that the equations describing a vector meson without sources permit a gauge transformation in which one of the potentials becomes zero, the other becomes equal to one of the fields. In the second part of the note the analogue of a Hertz potential is introduced and the relation to the fields obtained. *H. Feshbach.*

Fradkin, E. S. On the problem of the reaction of the self field of a charged particle. Nat. Res. Council Canada Tech. Translation TT-194, Ottawa, i+16 pp. (1951).

Translated from Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 20, 211–217 (1950); these Rev. 12, 66.

Guy, Roland. Existence des solutions de l'équation opératorielle d'évolution. C. R. Acad. Sci. Paris 233, 288–290 (1951).

The existence of solutions of Visconti's operatorial equation is demonstrated by a conventional method of successive approximations. *C. C. Torrance* (Annapolis, Md.).

Schützer, Walter, and Tiomno, J. On the connection of the scattering and derivative matrices with causality. Physical Rev. (2) 83, 249–251 (1951).

Kronig [Physica 12, 543–544 (1946)] has suggested that a "causality condition" should be used in the theory of the S-matrix according to which there should be no scattered wave before the incident wave reaches the scatterer. The authors analyze the consequences of this condition and show that it will be fulfilled if the S-matrix is represented by the relation $S(k)=\exp(-2ika)\cdot(1+ikR)/(1-ikR)$ where R is the derivative matrix (the reciprocal logarithmic derivative of the wave function). According to Wigner and Eisenbud [same Rev. (2) 72, 29–41 (1947)] the matrix R is a single-valued analytic function of the energy E; it is real for real E, and its poles are all on the real axis. Thus S can have poles only where $1-ikR=0$. It follows that the poles of $S(k)$ are either on the imaginary axis or in the lower half-plane, k is the wave number of the incident particle and a is the point at which the reciprocal logarithmic derivative is taken. *E. Gora* (Providence, R. I.).

Kamefuchi, Susumu. Note on the direct interaction between spinor fields. Progress Theoret. Physics 6, 175–181 (1951).

The divergences occurring in field theory of the β-decay type in which the coupling is quadrilinear in four spinor fields is considered. It is found that, in addition to divergences that can be removed by the customary mass renormalization arguments, there are infinite terms which describe the scattering of one fermion by another. These presumably cannot be removed by a reasonable renormalization procedure. *K. M. Case* (Ann Arbor, Mich.).

Roberts, K. V. On the quantum theory of the elementary particles. II. Quantum field dynamics. Proc. Roy. Soc. London. Ser. A. 207, 228–251 (1951).

Previous work [Roberts, same Proc. Ser. A. 204, 123–144 (1950); these Rev. 12, 465] on a covariant formulation of field theory following Weiss [ibid. 169, 102–119, 119–133 (1938)] is continued. Quantization is discussed in the Schrödinger, Heisenberg, and interaction representations. After proving the theory to be covariant it is shown that consistency depends on satisfying the Dirac integrability conditions. These can be fulfilled for any system which is described by a Lagrangian. After arriving at the Dyson expression for the S-matrix in terms of the interaction Hamiltonian it is indicated that it can be most simply evaluated with the use of the interaction Lagrangian. *K. M. Case* (Ann Arbor, Mich.).

Tonnelat, Marie-Antoinette. Étude du système formé par la réunion de deux corpuscules de Dirac. J. Phys. Radium (8) 12, 516–520 (1951).

This paper, dealing with a system of two interacting Dirac particles, is intended to complete and extend the work of E. Fermi and C. N. Yang [Physical Rev. (2) **76**, 1739–1743 (1949)] in which a meson is regarded as such a system. The starting point is similar to that of H. M. Moseley and N. Rosen [ibid. **80**, 177–181 (1950); these Rev. **12**, 574], but the subsequent development is different. It should be remarked that, after a certain point in the paper, the interaction function is tacitly assumed to be a constant, and solutions of the equations are obtained on this basis. The significance of the solutions is therefore not entirely clear.

N. Rosen (Chapel Hill, N. C.).

Arnous, E., und Zienau, S. Allgemeine Theorie der Dämpfungsphänomene für nicht-stationäre Prozesse. I. Grundlagen und Zusammenhang mit dem *S*-Matrix-Formalismus. Helvetica Phys. Acta 24, 279–295 (1951).

The authors intend to reformulate Heitler and Ma's quantum theory of radiation damping for discrete states [Proc. Roy. Irish Acad. Sect. A. **52**, 109–125 (1949); these Rev. **11**, 298] in such a way as to make possible: (1) a comparison with Dyson's *S*-matrix formalism; (2) an adaptation to positron theory; (3) an application of renormalization procedures to problems like the higher order radiative corrections to natural line widths. Only point (1) of this program is carried out in the present paper. The formalism is based on an exact time integration of the Schroedinger equation. After transformation to energy representation a "damping operator" is introduced whose diagonal elements represent the damping constants; its non-diagonal elements are the elements of the collision operator which is obtained by solving Heitler's integral equation. It appears that the damping formalism is equivalent with the *S*-matrix formalism but better adapted to the treatment of problems where energy is not conserved. As a first application to physical problems previously known formulas for line widths and line shifts are derived from the general formalism.

E. Gora (Providence, R. I.).

Heber, G. Zur Frage der magnetischen Momente der Nukleonen. I, II. Ann. Physik (6) 9, 151–168, 169–180 (1951).

The corrections to the magnetic moments of nucleons caused by their interactions with charged and neutral scalar and pseudoscalar mesons is calculated to second order in the coupling constants. Using Kallén's method [G. Kallén, Ark. Fys. **2**, 187–194 (1950); these Rev. **12**, 570], the equations of motion for the Heisenberg operators are integrated using a power series expansion. The resulting operators are then used to determine the corrections to the current operator. From this the changed magnetic moment is readily identified. Complete agreement with previous calculations is obtained and hence the same complete disagreement with experiment is found.

K. M. Case.

Thermodynamics, Statistical Mechanics

Agarwala, B. K., and Auluck, F. C. Statistical mechanics and partitions into non-integral powers of integers. Proc. Cambridge Philos. Soc. 47, 207–216 (1951).

As a generalization of various partition problems connected with statistical mechanics, the following one is attacked: Let q be a given positive integer, and σ a positive number not necessarily integral. Find the number $c(q, \sigma)$ of solutions $(a_1, a_2, a_3, \cdots)$ of the inequality

$$a_1^\sigma+a_2^\sigma+a_3^\sigma+\cdots\leqq q$$

subject to the condition $1\leqq a_1\leqq a_2\leqq\cdots$. Also, find the number $c^*(q, \sigma)$ of solutions of the same inequality subject to $1\leqq a_1<a_2<\cdots$. The appropriate thermodynamic assembly corresponding to the partition (generating) functions $p(q, \sigma)$, $p^*(q, \sigma)$ of $c(q, \sigma)$, $c^*(q, \sigma)$, respectively, is one obeying the Bose-Einstein or Fermi-Dirac statistics, and containing an indefinite number of similar particles, the energy levels of an individual particle being given by $\epsilon_r=\alpha r^\sigma$ $(r=1, 2, 3, \cdots)$. Appropriate expressions are obtained under special assumptions.

B. O. Koopman (New York, N. Y.).

Wild, E. On Boltzmann's equation in the kinetic theory of gases. Proc. Cambridge Philos. Soc. 47, 602–609 (1951).

By an ingenious device the author replaces Boltzmann's integro-differential equation by an integral equation which is equivalent to it, as far as differentiable solutions are concerned. In general, of course, the mean collision frequency ν of the molecules as a function of space, velocity, and time is not known. However, in certain cases, including that of Maxwellian molecules, it can be guessed in advance. Supposing ν given, the author shows that the solution of the general initial value problem for the Boltzmann equation, if it exists at all, can be found by a formally simple iterative process which can be represented symbolically by the scheme

$$f_1=f_0+S(f_0\bigcirc f_0),\quad f_{r+1}=f_0+S(f_r\bigcirc f_r),$$

where f is a certain function determined by the initial data, "$\bigcirc$" is a certain composition rule defined in terms of the collision cross-section, and S is a certain integral operator defined in terms of the assumed value for ν. Proof that the iterative process converges depends upon assuming the existence of a solution to the integral equation with ν fixed. The solution $f=\lim f_r$ is not in general a solution of Boltzmann's equation unless f turns out to be related to ν in the proper way. In the case of Maxwellian molecules, for a spatially homogeneous initial velocity distribution the calculations can be carried out explicitly, and the author shows that if a solution exists at all, it is given by a rather simple series. In this case follows also a much stronger result, viz., if the initial homogeneous distribution is majorized by a Maxwellian distribution, then a solution of the corresponding initial value problem for the Boltzmann equation for Maxwellian molecules exists in a finite time interval and is given by the author's formula. This solution, also homogeneous, yields a steady mass density. Finally, the author discusses conditions under which a homogeneous solution, presumed to exist for all future times, approaches a Maxwellian one. He shows that if the initial distribution is near to a Maxwellian one in a certain sense which he defines, then the corresponding solution approaches the Maxwellian one uniformly.

The difficulty of the problem treated by the author is well known. Despite a brilliant attack by Hilbert [Math. Ann. **72**, 562–577 (1912)], apart from a single very special theorem of Carleman [Acta Math. **60**, 91–146 (1933)], the author's result for Maxwellian molecules, incomplete as it is, constitutes the only positive progress made in the analytical theory of Boltzmann's equation up to this time. Unfortunately the paper is written in such a condensed and crabbed style that it is difficult to find out what is actually done.

C. A. Truesdell (Bloomington, Ind.).

Scheidegger, A. E., and McKay, C. D. Quantum statistics of fields. Physical Rev. (2) **83**, 125–131 (1951).

✱Bogolyubov, N. N. Problemy dinamičeskoĭ teorii v statističeskoĭ fizike. [**Problems of Dynamical Theory in Statistical Physics**]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1946. 120 pp.

I. Expansion in powers of a small parameter in the theory of statistical equilibrium. II. The kinetic equations in classical mechanics. *Table of contents.*

Dutta, M. On a treatment of imperfect gas after Fermi's model. III. Proc. Nat. Inst. Sci. India **17**, 27–37 (1951).

This paper develops, by an exclusion principle, a statistical theory of imperfect gases subject to a general field of force. An equation of state of Dieterici's form is obtained.

C. C. Torrance (Annapolis, Md.).

Zoller, K. Zur Stuktur des Verdichtungsstosses. Z. Physik **130**, 1–38 (1951).

The early investigators of the structure of strong shock waves recognized the steepness of velocity and temperature gradients in it and pointed out the inadequacy of the Navier-Stokes equations. The obvious improvement is the inclusion of the second order terms, the so-called "Burnett equations," in the analysis. Recent investigations by L. H. Thomas [J. Chem. Phys. **12**, 449–453 (1944)] and C. S. Wang-Chang [University of Michigan, Dept. of Engineering Report **UMH-3-F(APL/JHU CM-503)** (1948)] have shown, however, such an approach will be fruitless as the underlying Enskog-Chapman expansion procedure for the Boltzmann equation is not convergent for the problem of strong shock. This led H. M. Mott-Smith [Physical Rev. (2) **82**, 885–892 (1951); these Rev. **12**, 891] to a different formulation of the problem avoiding the Enskog-Chapman expansion procedure. His result, although only approximate, is thus more reliable. The paper under review however uses again the expansion procedure and the method of Burnett. He stops at the second order terms, and shows first that the second order terms are comparable or larger than the first order terms (Navier-Stokes terms) for shock pressure ratio greater than 4. By calculating the behavior of solutions near the conditions far ahead and far downstream of the shock (equilibrium states), the author also discovers other anomalies for strong shocks which can be attributed to the failure of the Enskog-Chapman procedure. The author finally calculates the "thickness" of the shock and shows that the effect of the second order terms is to increase it, agreeing with the results of other investigators. *H. Tsien.*

Kirkwood, John G., Maun, Eugene K., and Alder, Berni J. Radial distribution functions and the equation of state of a fluid composed of rigid spherical molecules. J. Chem. Phys. **18**, 1040–1047 (1950).

Kirkwood [J. Chem. Phys. **3**, 300–313 (1935)] has derived an integral-differential equation for the radial distribution function of spherically symmetrical molecules about a specified molecule in a classical system of many identical interacting molecules by assuming that the potential of average force acting on a set of three molecules is the sum of the potentials of average force acting on the three pairs in the set (this assumption has been called the superposition principle). A similar integral-differential equation has been derived by Born and Green [Proc. Roy. Soc. London. Ser. A. **188**, 10–18 (1946); these Rev. **9**, 402] with the assumption that the position distribution function of a set of three molecules is the product of the three pair distribution functions of the pairs in the set.

In this paper the authors have solved both the Kirkwood and Born-Green equations by punch card methods to obtain the radial distribution function of a system of hard spheres with no attractive forces. The equation of state is derived from the radial distribution function by employing the virial theorem. The results are compared with those based on the free volume theory of liquids. The superposition principle implies that an ordered or crystalline phase (with a sharp phase transition) exists at sufficiently high densities even in the absence of attractive forces between the molecules. *E. W. Montroll.*

Burton, W. K., Cabrera, N., and Frank, F. C. The growth of crystals and the equilibrium structure of their surfaces. Philos. Trans. Roy. Soc. London. Ser. A. **243**, 299–358 (1951).

The first two parts of this paper present a theory of the growth of a crystal from a vapor or solution in which the crystal grows in the form of spiral pyramids on the various faces, i.e., these faces are not flat but have defects involving a slight screw component for the molecular arrangement. In the case in which the supersaturation is not large this theory replaces the classical nucleation theory of growth. Mathematically, a flow of molecules absorbed in such a surface is studied statistically by straightforward thermodynamic methods. Part III contains a discussion of crystal surfaces at and near saturation. Previous studies of the shape of the critical nucleus in this region are extended but the result still indicates the inadequacy of this theory at moderate supersaturations. The authors also give a statistical study of steps in a crystal surface and obtain estimates of the number of kinks in such a step. However, in the fourth part, the authors consider the crystal surface at equilibrium as a problem in cooperative phenomena. Thus, if the surface is supposed to be in two layers, a change in level of the surface results in a contribution to the potential energy and this problem can be solved by the Onsager method of obtaining the partition function for a two-dimensional lattice. The result is that, at sufficiently low temperatures, steps will not occur in the crystal surface but above the critical point the surface will become rough. *F. J. Murray.*

Oguchi, Takehiko. Statistics of the three-dimensional ferromagnet. I. J. Phys. Soc. Japan **5**, 75–81 (1950).

High and low temperature approximations for the partition function of a simple cubic lattice with Ising interaction are derived by the perturbation method of Ashkin and Lamb [Physical Rev. (2) **64**, 159–178 (1943)]. The approximations are not carried out as far as those of Trefftz [Z. Physik **127**, 371–380 (1950); these Rev. **12**, 468] but they are consistent with hers to the same approximation.

E. Montroll (College Park, Md.).